# RANDOM WALKS ON GROUPS

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Abstract. These are notes for a topics course I taught in Spring 2025 at the University of Wisconsin-Madison.

There are probably many typos, imprecise references, and other mistakes. Any comments are very welcome!

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*Date*: December 26, 2025.

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## 1. Introduction

Given a probability measure $\mu$ on a group $G$, we will study asymptotic behavior of the random variables

$$g_1 g_2 \cdots g_n$$

where the $g_i$ are independent identically distributed elements of $G$ each with distribution $\mu$.

In the rest of this section we will briefly outline what is covered in these lecture notes. Much of our initial discussion will be based on the books of Lalley [Lal23] and Benoist–Quint [BQ16].

**Part # 1: The toolbox.** In the first part of the course we will develop a tool box for studying random products. This includes introducing some important theorems from ergodic theory and developing a theory of stationary measures.

**Part # 2: Random walks on linear groups.** In Part #2 we consider a probability measure $\mu$ on $\mathsf{SL}(d,\mathbb{R})$ and let $g_1, g_2, \dots$ be independent identically distributed elements each with distribution $\mu$. We will be primarily interested in studying the asymptotics of the singular values of

$$g_1 \cdots g_n$$

as $n \to +\infty$.

Recall that every $g \in \mathsf{SL}(d,\mathbb{R})$ has a singular value decomposition $g = ka\ell$ where $k, \ell \in \mathsf{SO}(d)$,

$$a = \begin{pmatrix} \sigma_1(g) & & \\ & \ddots & \\ & & \sigma_d(g) \end{pmatrix},$$

and $\sigma_1(g) \geq \sigma_2(g) \geq \cdots \geq \sigma_d(g)$ are the singular values of $g$.

**Warning!** The singular values are unique, but the orthogonal matrices in the singular value decomposition are not. For instance, let

$$\mathsf{M} := \{\mathrm{diag}(\pm 1, \dots, \pm 1)\} \cap \mathsf{SL}(d, \mathbb{R}).$$

Now, if $g = ka\ell$ is a singular value decomposition, then so is $g = (km)a(m\ell)$ for all $m \in \mathsf{M}$.

Let $\|g\| = \sigma_1(g)$ denote the operator norm of an element $g \in \mathsf{SL}(d, \mathbb{R})$. As a corollary to the Kingman Ergodic Theorem, we will prove the following.

**Theorem 1.1** (see Theorem 5.5 below)**.** *If*

$$\int_{\mathsf{SL}(d,\mathbb{R})} \log \|g\| \, d\mu(g) < +\infty,$$

*then for* $1 \leq p \leq d-1$*, the number*

$$\lambda_p(\mu) := \lim_{m\to\infty} \frac{1}{m} \int_G \log \sigma_p(g) d\mu^{*m}(g)$$

*is called the* $p^{th}$***-Lyapunov exponent of*** $\mu$ *and satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \log \sigma_p(g_1 \cdots g_n) = \lambda_p(\mu) = \lim_{n\to\infty} \frac{1}{n} \log \sigma_p(g_n \cdots g_1)$$

*almost surely.*

By definition one has

$$\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_d(\mu)$$

and we will show when the support of the measure generates a Zariski dense group, there is strict inequalities between each Lyapunov exponent.

A subgroup $\mathsf{H} \subset \mathsf{SL}(d, \mathbb{R})$ is *Zariski dense* if, viewing $\mathsf{SL}(d, \mathbb{R}) \subset \mathbb{R}^{d^2}$, whenever $P : \mathbb{R}^{d^2} \to \mathbb{R}$ is a polynomial vanishing on $\mathsf{H}$, then $P$ also vanishes on $\mathsf{SL}(d, \mathbb{R})$. Being Zariski dense is quite mild and is, in some sense, a generic condition.

**Example 1.2.** We can view $\mathsf{SL}(d, \mathbb{R})$ is a smooth submanifold of $\mathbb{R}^{d^2}$. If $\lambda$ is a probability measure on $\mathsf{SL}(d, \mathbb{R})$ which is induced by a smooth volume form and $X, Y$ are independent identically distributed elements of $\mathsf{SL}(d, \mathbb{R})$ each with distribution $\lambda$, then

$$\Pr(X, Y \text{ generate a Zariski dense subgroup}) = 1.$$

For details see [Bre08, Lemma 6.8].

**Theorem 1.3** (Gol'dsheĭd–Margulis [GM89])**.** *If*

- $\int_{\mathsf{SL}(d,\mathbb{R})} \log \|g\| \, d\mu(g) < +\infty$ *and*
- *the group generated by* $\mathrm{supp}(g)$ *is Zariski dense,*

*then*

$$\lambda_1(\mu) > \lambda_2(\mu) > \cdots > \lambda_d(\mu).$$

**Part #3: Random Walks on Gromov Hyperbolic Spaces.** In this part we will consider random walks on the isometry group $\mathsf{Isom}(X)$ of a proper metric space $X$. In particular, we consider a probability measure $\mu$ on $\mathsf{Isom}(X)$ and let $g_1, g_2, \dots$ be independent identically distributed elements each with distribution $\mu$. We will be primarily interested in studying the asymptotics of the orbits of

$$g_1 \cdots g_n(x_0)$$

as $n \to +\infty$, where $x_0 \in X$ is some basepoint.

As a corollary to the Kingman Ergodic Theorem, we will prove the following.

**Theorem 1.4** (see Theorem 5.3 below)**.** *If $x_0 \in X$ and*

$$\int_{\mathsf{Isom}(X)} \mathrm{d}_X(x_0, gx_0) d\mu(g) < +\infty,$$

*then there exists a number $L_\mu \geq 0$, called the **linear drift of** $\mu$, such that*

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{d}_X(x_0, g_1 \cdots g_n(x_0)) = L_\mu$$

*almost surely.*

The main goal of this part of the notes is to prove that linear drift is positive when $X$ is a proper geodesic Gromov hyperbolic metric space and the group generated by the support of probability measure is "non-elementary."

So, let $X$ be a proper geodesic Gromov hyperbolic metric space. Then $X$ has a natural compactification $\overline{X}^{\mathcal{G}} = X \sqcup \partial_{\mathcal{G}} X$ called the Gromov compactification. The isometry group extends continuously to this compactification and a **semigroup** $H \subset \mathsf{Isom}(X)$ is *non-elementary* if

(1) There are no $H$-invariant subsets in $\partial_{\mathcal{G}} X$ with cardinality 1 or 2.
(2) Every (some) orbit of $H$ in $X$ is unbounded.

We will prove the following.

**Theorem 1.5** (see Theorem 19.1 below)**.** *Suppose the semigroup generated by* $\mathrm{supp}(\mu)$ *is non-elementary. Then:*

*(1) If $x_0 \in X$, then*

$$\lim_{n\to\infty} g_1 \cdots g_n(x_0)$$

*exists and is in $\partial_{\mathcal{G}} X$ almost surely.*

*(2) If $\int_{\mathsf{Isom}(X)} \mathrm{d}(o, g(x_0)) d\mu(g) < +\infty$ for some (any) $x_0 \in X$, then the linear drift is positive.*

For more properties of random walks in this context, see Maher–Tiozzo [MT18] (who prove much more and consider non-proper Gromov hyperbolic spaces).

The proof we provide for Theorem 1.5 uses an integral formula for the linear drift due to Karlsson–Ledrappier [KL11] and is very similar to the proof of Theorem 1.3.

**Part #4: Random walks on finitely generated groups.** In Parts 4 through 6, we consider a probability measure $\mu$ on a finitely generated group $\Gamma$. As before, we let $g_1, g_2, \dots$ be independent identically distributed elements each with distribution $\mu$. The random variable

$$g_1 \cdots g_n$$

describes the $n$-th step of a random walk on $\Gamma$ starting at the identity with step distribution $\mu$.

The main result of this part proves that when $\Gamma$ is non-amenable, the random walk always has positive linear drift.

**Theorem 1.6** (see Theorem 26.1 below)**.** *Suppose* $\mathrm{d}_S$ *is a word metric on* $\Gamma$ *and* $\mu$ *induces an irreducible random walk. If* $\Gamma$ *is non-amenable, then the linear drift*

$$L_\mu := \lim_{n\to\infty} \frac{\mathrm{d}_S(\mathrm{id}, g_1 \cdots g_n)}{n}$$

*is positive (possibly equal to* $+\infty$*).*

*Remark* 1.7. Notice that we do not assume that $\int \mathrm{d}_S(\mathrm{id}, g) d\mu(g) < +\infty$ in this theorem.

We also introduce the Green metric associated to a random walk and establish some of its basic properties (see Section 28).

**Part #5: The Martin Boundary.** In this part we introduce the Martin compactification $\overline{\Gamma}^M = \Gamma \sqcup \partial_M \Gamma$ associated to to a probability measure $\mu$ on a finitely generated group $\Gamma$ where the associated random walk is irreducible and transient. The two main results we establish are

(1) Every positive harmonic function can be described by an integral over the Martin boundary.
(2) A random walk almost surely converges to a point in the Martin boundary.

(see Theorems 32.1 and 29.3)

**Part #6: The Martin Boundary of a word hyperbolic group.** In this part, we consider a word hyperbolic group $\Gamma$ with a finitely supported probability measure $\mu$ whose support generates $\Gamma$ as a semigroup. In this case, we prove that the Gromov boundary coincides with the Martin boundary.

This result is originally due to Ancona [Anc88, Anc87], but the proof we provide is based on recent work of Gekhtman–Gerasimov–Potyagailo–Yang [GGPY21] which extends Ancona's results to relatively hyperbolic groups.

**Dependencies.**

- Part 2 uses the main results from Part 1.
- Part 3 is independent of Part 2 and uses the main results from Part 1.
- Part 4 is essentially independent from Parts 1, 2, and 3, and only uses the notation introduced in Section 3.
- Part 5 requires Part 4, except for Section 26 and the discussion of the Green metric in Section 28.
- Part 6 requires almost all of Parts 4 and 5.

# Part 1. The toolbox

## 2. The Birkhoff ergodic theorem

In this section we define ergodicity and state the Birkhoff ergodic theorem, for more background on ergodic theory see the discussion in Lalley's book [Lal23] or standard texts in ergodic theory [EW11, Wal82].

For the rest of the section fix a measurable space $(X, \mathcal{A})$ and a measurable map $T : X \to X$ (i.e. $T^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$). Given a measure $\lambda$ on $(X, \mathcal{A})$ the *push-forward* of $\lambda$ under $T$ is the measure

$$T_*\lambda(A) = \lambda(T^{-1}A).$$

**Definition 2.1.**

- A measure $\lambda$ is $T$-*invariant* if $T_*\lambda = \lambda$.
- A **finite** $T$-invariant measure $\lambda$ is *ergodic* if every measurable set $A$ with $T^{-1}(A) = A$ has either zero or full measure.

There are a number of basic equivalences.

**Observation 2.2.** Suppose $\lambda$ is a finite $T$-invariant measure. Then the following are equivalent:

(1) $\lambda$ is ergodic.
(2) Every measurable subset $A \subset X$ with $\lambda(A \Delta T^{-1}A) = 0$ either has zero or full measure.
(3) Every measurable function $f : X \to \mathbb{R}$ with $f \circ T = f$ almost everywhere is constant almost everywhere.

*Remark* 2.3. $A \Delta B = (A - B) \cup (B - A)$ is the symmetric difference.

*Remark* 2.4. When the measure is fixed, we often say that a measurable function $f : X \to \mathbb{R}$ is $T$-*invariant* if $f \circ T = f$ almost everywhere and that a measurable set $A \in \mathcal{A}$ is $T$-*invariant* if $A \Delta T^{-1}A$ has measure zero.

*Proof.* (1) $\Rightarrow$ (2). Let

$$A' := \{x : T^j x \in A \text{ for infinitely many } j\} = \bigcap_{m \geq 1} \bigcup_{j \geq m} T^{-j}A.$$

Then $T^{-1}A' = A'$ and hence $A'$ either has full or zero measure. Thus it suffices to show that $\lambda(A' \Delta A) = 0$ (which implies that $\lambda(A) = \lambda(A')$).

First notice that

$$\lambda(A \Delta T^{-j}A) \leq \sum_{i=0}^{j-1} \lambda(T^{-i}A \Delta T^{-(i+1)}A) = \sum_{i=0}^{j-1} \lambda(T^{-i}(A \Delta T^{-1}A)) = \sum_{i=0}^{j-1} \lambda(A \Delta T^{-1}A) = 0.$$

Then

$$\lambda(A \Delta \cup_{j \geq m} T^{-j}A) \leq \sum_{j=m}^{\infty} \lambda(A \Delta T^{-j}A) = 0.$$

Finally, using monotonicity,

$$\lambda(A \Delta A') = \lim_{m \to \infty} \lambda(A \Delta \cup_{j \geq m} T^{-j}A) = 0.$$

(2) $\Rightarrow$ (3). For $c \in \mathbb{R}$, let $E_c := \{x \in X : f(x) \geq c\}$. Then, since $f \circ T = f$ almost everywhere, $\lambda(E_c \Delta T^{-1}E_c) = 0$ and so $E_c$ either has zero or full measure. Thus $f = \sup\{c : E_c \text{ has full measure}\}$ almost everywhere.

(3) $\Rightarrow$ (1). If $T^{-1}A = A$, then $1_A \circ T = 1_A$. Hence $1_A$ is constant almost everywhere, which implies that $A$ either has zero or full measure. $\square$

Ergodic systems have the following remarkable property.

**Theorem 2.5** (Birkhoff Ergodic Theorem)**.** *Suppose $T : (X, \lambda) \to (X, \lambda)$ is ergodic. If $f \in L^1(X, \lambda)$, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \frac{1}{\lambda(X)} \int f d\lambda$$

*for almost every $x \in X$.*

*Proof.* For a proof of Birkhoff's theorem see any of [Lal23, EW11, Wal82]. □

Birkhoff's theorem states that for ergodic systems, the average of $f$ along a generic orbit coincides with the average value of $f$.

Using Birkhoff's theorem we will provide a geometric description of the ergodic probability measures (it is also possible to prove this directly). Let $\mathcal{P}_T(X)$ denote the set of $T$-invariant probability measures, which is a convex subset of the space of all probability measures.

**Proposition 2.6.** *A $T$-invariant probability measure $\lambda$ is ergodic if and only if it is an extreme point of $\mathcal{P}_T(X)$, i.e. if $\lambda = t\lambda_1 + (1-t)\lambda_2$ where $0 < t < 1$ and $\lambda_1, \lambda_2 \in \mathcal{P}_T(X)$, then $\lambda = \lambda_1 = \lambda_2$.*

*Proof.* ($\Leftarrow$). Suppose for a contradiction that $T^{-1}A = A$ for some measurable set with $\lambda(A) \in (0, 1)$. Let $\lambda_1 = \frac{1}{\lambda(A)}\lambda(A \cap \cdot)$ and $\lambda_2 = \frac{1}{\lambda(A^c)}\lambda(A^c \cap \cdot)$. Then

$$\lambda = \lambda(A)\lambda_1 + (1 - \lambda(A))\lambda_2$$

and both measures are $T$-invariant. So $\lambda = \lambda_1 = \lambda_2$. Since $\lambda_1, \lambda_2$ are singular this is impossible.

($\Rightarrow$). Suppose $\lambda = t\lambda_1 + (1-t)\lambda_2$ where $0 < t < 1$ and $\lambda_1, \lambda_2 \in \mathcal{P}_T(X)$. Since each $\lambda_i$ is absolutely continuous with respect to $\lambda$, any set with full or zero $\lambda$-measure has full or zero $\lambda_i$-measure. Thus each $\lambda_i$ is ergodic.

Fix a measurable subset $A \subset X$, then applying Birkhoff's theorem we have

$$\frac{1}{N} \sum_{j=0}^{N-1} 1_A(T^j x) \to \lambda_i(A)$$

for $\lambda_i$-almost every $x$. However, applying Birkhoff's theorem we also have

$$\frac{1}{N} \sum_{j=0}^{N-1} 1_A(T^j x) \to \lambda(A)$$

for $\lambda$-almost every $x$. Since $\lambda_i \ll \lambda$, we can fix $x$ satisfying both conditions. Hence $\lambda_i(A) = \lambda(A)$. Thus $\lambda_i = \lambda$.

□

2.1. **Existence of ergodic measures.** Suppose $Y$ is a compact metrizable space and $T : Y \to Y$ is continuous. We will show that there exists an ergodic probability measure. Let $\mathcal{P}(Y)$ denote the space of Borel probability measures on $Y$ with the weak-$*$ topology, i.e.

$$\mu_n \to \mu \iff \int f d\mu_n \to \int f d\mu$$

for all continuous functions $f : Y \to \mathbb{R}$. With this topology, $\mathcal{P}(Y)$ is compact.

**Proposition 2.7.** *If $y_0 \in Y$ and $\mu_n := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j y_0}$, then any weak-$*$ limit point of $\{\mu_n\}$ is $T$-invariant.*

*Proof.* Suppose $\mu_{n_j} \to \mu$. Then

$$T_*\mu_{n_j} = \frac{1}{n_j}\sum_{j=0}^{n_j-1} \delta_{T^{j+1}y_0} = \mu_{n_j} + \frac{1}{n_j}\delta_{T^{n_j}y_0} - \frac{1}{n_j}\delta_{y_0}.$$

Taking $j \to \infty$ we get $T_*\mu = \mu$. □

**Corollary 2.8.** *There exists a probability measure on $Y$ which is ergodic with respect to $T$.*

*Proof.* Let $\mathcal{P}_T(Y)$ denote the $T$-invariant measures. Then $\mathcal{P}_T(Y)$ is a non-empty convex subset of a reasonable vector space (the dual of continuous functions on $Y$) and hence by the Krein–Milman theorem $\mathcal{P}_T(Y)$ has extreme points. So by Proposition 2.6, there exists a probability measure on $Y$ which is ergodic with respect to $T$. □

2.2. **Examples.** We just mention a few ergodic systems (which will never be discussed again in this class).

(1) Let $\mathbb{S}^1 \subset \mathbb{C}$ be the unit circle and let $\lambda$ be the pushforward of the Lebesgue measure under the map $\theta \in [0, 2\pi] \mapsto e^{i\theta} \in \mathbb{S}^1$. Then let $T : \mathbb{S}^1 \to \mathbb{S}^1$ be rotation by a number $2\pi\alpha$ with $\alpha$ irrational, i.e. $T(e^{i\theta}) = e^{i(\theta+2\pi\alpha)}$. Then $T$ is ergodic with respect to $\lambda$.

(2) Let $\mathbb{T} = \mathbb{R}^2 / \mathbb{Z}^2$ be the Torus and let $\lambda$ be the pushforward of the Lebesgue measure restricted to $[0,1]\times[0,1]$ under the map $(x, y) \mapsto (x, y)+\mathbb{Z}^2$. Given $g \in \mathsf{SL}(2, \mathbb{Z})$ define $T_g : \mathbb{T} \to \mathbb{T}$ by

$$T_g((x, y) + \mathbb{Z}^2) = g(x, y) + \mathbb{Z}^2\,.$$

If $g$ has no eigenvalue with modulus one, then $T_g$ is ergodic with respect to $\lambda$.

In both cases, one verifies ergodicity using the Fourier series!

## 3. The mathematical setup for studying random walks

Delaying some definitions, our usual setup/notation is the following:

- $G$ is a second countable locally compact Hausdorff topological group.
- $\mu$ is a Borel probability measure on $G$.
- $\Omega = G^{\mathbb{N}}$ (i.e. sequences in $G$), which we can interpret as the space of all possible random products. We use $\mathbf{g} = (g_1, g_2, \dots)$ to denote an element of $\Omega$.
- $\mathcal{F}$ is the product sigma algebra on $\Omega$.
- $\beta = \mu^{\otimes \mathbb{N}}$ is the product probability measure on $\Omega$.
- $w_n : \Omega \to G$ is the map $w_n(\mathbf{g}) = g_1 \cdots g_n$.
- $\mu^{*n}$ is the $n$-fold convolution of $\mu$.

We now explain the words used above.

**Topological groups.** A *topological group* $G$ is a group which is also a topological space where multiplication $(g, h) \in G \times G \mapsto gh \in G$ and inversion $g \in G \mapsto g^{-1} \in G$ are both continuous.

The conditions "second countable locally compact Hausdorff" imply that the topology is metrizable and hence every Borel measure is regular.

We will almost always consider the case when $G$ is a countable group with the discrete topology or when $G = \mathsf{SL}(d, \mathbb{R})$, which has the subspace topology from the natural inclusion $\mathsf{SL}(d, \mathbb{R}) \subset \mathbb{R}^{d^2}$.

**The product sigma-algebra on $\Omega = G^{\mathbb{N}}$.** A subset $C \subset \Omega$ is called a *cylinder* if

$$C = \prod C_i,$$

each $C_i$ is a Borel measurable subset of $G$, and the set $\{i : C_i \neq G\}$ is finite. The product sigma algebra on $\Omega$ is the smallest sigma algebra containing all of the cylinders.

Since $G$ is second countable, the product sigma-algebra on $G^{\mathbb{N}}$ coincides with the Borel sigma algebra with respect to the product topology on $G^{\mathbb{N}}$.

**The product probability measure on $\Omega = G^{\mathbb{N}}$.** The measure $\beta = \mu^{\otimes \mathbb{N}}$ is the unique probability measure (defined on the product sigma algebra) where $\beta(C) = \prod \mu(C_i)$ for every cylinder $C = \prod C_i$.

**Convolutions of measures.** The *convolution*, denoted $\lambda * \nu$, of two measures $\lambda, \nu$ on $G$, is a measure on $G$ defined by

$$\lambda * \nu = \int_G g_* \nu d\lambda(g),$$

i.e. for a measurable set

$$\lambda * \nu(A) = \int_G g_* \nu(A) d\lambda(g) = \int_G \nu(g^{-1}A) d\lambda(g).$$

The convolution of two measures is the push-forward under multiplication of their product.

**Observation 3.1.** If $m : G \times G \to G$ is the multiplication map $m(g, h) = gh$, then $m_*(\lambda \otimes \nu) = \lambda * \nu$.

*Proof.* Notice that

$$\begin{aligned}\lambda * \nu(A) &= \int_G \nu(g^{-1}A) d\lambda(g) = \int_G \int_G 1_{g^{-1}A}(h) d\nu(h) d\lambda(g) = \int_G \int_G 1_A(gh) d\nu(h) d\lambda(g) \\ &= (\lambda \otimes \nu)(\{(g, h) : gh \in A\}) = (\lambda \otimes \nu)(m^{-1}A) = m_*(\lambda \otimes \nu)(A).\end{aligned}$$

□

As an immediate consequence we have the following.

**Observation 3.2.** For an integrable function

$$\int_G f d(\lambda * \nu) = \int_G \int_G f(gh) d\lambda(g) d\nu(h).$$

*Proof.* Note that

$$\begin{aligned}\int_G f d(\lambda * \nu) &= \int_G f dm_*(\lambda \otimes \nu) = \int_{G \times G} (f \circ m) d(\lambda \otimes \nu) \\ &= \int_G \int_G f(gh) d\lambda(g) d\nu(h).\end{aligned}$$

□

The $n$-fold convolution of $\mu$, denoted by $\mu^{*n} = \mu * \cdots * \mu$ is of particular importance for studying random products. Recall $w_n : \Omega \to G$ is the map $w_n(\mathbf{g}) = g_1 \cdots g_n$.

**Observation 3.3.** For all measurable subsets $A \subset G$,

$$\mu^{*n}(A) = \Pr(w_n \in A).$$

Equivalently, $(w_n)_*\beta = \mu^{*n}$.

*Proof.* By definition,

$$\mu^{*n}(A) = \int_G 1_A d\mu^{*n} = \int_G \cdots \int_G 1_A(g_1 \cdots g_n) d\mu(g_1) \cdots d\mu(g_n).$$

Let $B := \{\mathbf{g} : w_n \in A\}$. Then $B = B' \times G^{\mathbb{N}}$ where $B' \subset G^n$. So by definition,

$$\Pr(w_n \in A) = \beta(B) = (\mu^{\otimes n})(B') = \int_G \cdots \int_G 1_A(g_1 \cdots g_n) d\mu(g_1) \cdots d\mu(g_n).$$

□

## 4. Ergodicity of the shift map

Let $G$, $\mu$, $\Omega = G^{\mathbb{N}}$, $\beta = \mu^{\otimes \mathbb{N}}$, etc. be as usual. Define $T : \Omega \to \Omega$ by

$$T(g_1, g_2, g_3, \dots) = (g_2, g_3, \dots).$$

This transformation will be very useful for studying random products.

**Theorem 4.1.** *$T : (\Omega, \beta) \to (\Omega, \beta)$ is ergodic.*

Since $T$ is continuous, $T$ is (Borel) measurable. Also, if $C = C_1 \times C_2 \times \cdots$ is a cylinder, then

$$T_*\beta(C) = \beta(T^{-1}(C)) = \beta(G \times C_1 \times C_2 \times \cdots) = \prod \mu(C_i).$$

Since $\beta$ is the unique probability measure with this property, we must have $T_*\beta = \beta$. So $\beta$ is $T$-invariant.

We provide two proofs of ergodicity.

*Proof of Ergodicity #1.* To show that $T$ is ergodic, fix a measurable set $E$ with $T^{-1}(E) = E$. Since $\beta$ is a probability measure, to show that $E$ has either zero or full measure it suffices to show that $\beta(E) \leq \beta(E)^2$.

Fix $\epsilon > 0$. By standard approximation results (see Corollary A.2.4 in [Lal23]), we can find a finite union of cylinders $A$ such that

$$\int |1_E - 1_A| \, d\beta < \epsilon,$$

which implies that $|\beta(E) - \beta(A)| < \epsilon$.

Since $A$ is a finite union of cylinders, there exists $n \in \mathbb{N}$ and $A_0 \subset G^n$ such that $A = A_0 \times G \times G \cdots$. Then

$$T^{-n}(A) = G^n \times A_0 \times G \times G \cdots$$

and so

$$\int 1_A 1_{T^{-n}(A)} d\beta = \beta(A \cap T^{-n}(A)) = \beta(A_0 \times A_0 \times G \times G \times \cdots) = \mu^{\otimes n}(A_0)^2 = \beta(A)^2.$$

Then

$$\begin{aligned}\beta(E) = \beta(E \cap T^{-n}(E)) &= \int 1_E 1_{T^{-n}(E)} d\beta \\ &= \int 1_A 1_{T^{-n}(A)} d\beta + \int (1_E - 1_A) 1_{T^{-n}(A)} d\beta + \int 1_E (1_{T^{-n}(E)} - 1_{T^{-n}(A)}) d\beta \\ &\leq \beta(A)^2 + \int |1_E - 1_A| \, d\beta + \int |1_E \circ T^n - 1_A \circ T^n| \, d\beta \\ &< (\beta(E) + \epsilon)^2 + 2\epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, $\beta(E) \leq \beta(E)^2$. Hence $E$ has either zero or full measure.

Thus $T$ is ergodic. □

*Proof of Ergodicity #2.* For $n \geq 0$, let $\mathcal{H}_n \subset L^2 = L^2(\Omega, \beta)$ be the $L^2$-functions that depend only on $g_1, \ldots, g_n$. Then each $\mathcal{H}_n$ is a closed subspace, $\cup_n \mathcal{H}_n$ is dense in $L^2$, and the orthogonal projection $P_n : L^2 \to \mathcal{H}_n$ is given by

$$P_n(f)(g_1, \ldots, g_n) = \int_\Omega f(g_1, \ldots, g_n, \mathbf{h}) d\beta(\mathbf{h}).$$

Now fix a measurable set $E \subset \Omega$ with $T^{-1}E = E$. Then $1_E \circ T = 1_E$. Let $f_n = P_n(1_E)$. Then, since $\cup_n \mathcal{H}_n$ is dense in $L^2$, we have $f_n \to 1_E$ in $L^2$.

Since $1_E \circ T = 1_E$ we have

$$\begin{aligned}f_n(g_1, \ldots, g_n) &= \int_\Omega 1_E(g_1, \ldots, g_n, \mathbf{h}) d\beta(\mathbf{h}) = \int_\Omega 1_E(g_2, \ldots, g_n, \mathbf{h}) d\beta(\mathbf{h}) \\ &= \int_\Omega 1_E(g_3, \ldots, g_n, \mathbf{h}) d\beta(\mathbf{h}) = \cdots = \int_\Omega 1_E(\mathbf{h}) d\beta(\mathbf{h}) = \beta(E)\end{aligned}$$

almost everywhere. Since $f_n \to 1_E$ in $L^2$ we then have $1_E = \beta(E)$ almost everywhere. Hence $E$ has either zero or full measure.

□

4.1. **Strong law of large numbers.** As a consequence of the ergodicity of the shift space, we can prove the strong law of large numbers.

**Theorem 4.2** (Strong law of large numbers)**.** *Let $G = (\mathbb{R}, +)$ and let $\mu$ be a probability measure on $G$ with $\int |x| \, d\mu(x) < +\infty$. If $g_1, g_2, \ldots$ are independent and identically distributed with distribution $\mu$, then*

$$\frac{g_1 + \cdots + g_n}{n} \to \int_{\mathbb{R}} x d\mu(x)$$

*almost surely.*

*Proof.* Define $f : \Omega \to \mathbb{R}$ by $f(\mathbf{g}) = g_1$. Then

$$\int_\Omega |f| \, d\beta = \int_{\mathbb{R}} |x| \, d\mu(x) < +\infty.$$

So by the ergodicity of the shift map,

$$\frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j(\mathbf{g}) \to \int_\Omega f d\beta$$

almost surely. The left hand side is

$$\frac{1}{n}\sum_{j=0}^{n-1} f\circ T^j(\mathbf{g}) = \frac{g_1+\cdots+g_n}{n}$$

and the right hand side is

$$\int_\Omega f d\beta = \int_{\mathbb{R}} x d\mu(x). \qquad \square$$

## 5. Kingman's subadditive ergodic theorem and applications

To fully exploit the ergodicity of the shift space, we will need a generalization of the Birkhoff ergodic theorem.

Given a map $T : X \to X$, a sequence of functions $f_n : X \to \mathbb{R}$ is *subadditive (with respect to $T$)* if

$$f_{n+m} \le f_n + f_m \circ T^n$$

for all $n, m \in \mathbb{N}$.

**Theorem 5.1** (Kingman Ergodic Theorem)**.** *Suppose $T : (X, \lambda) \to (X, \lambda)$ is ergodic and $\{f_n\} \subset L^1(X, \lambda)$ is a subadditive sequence. Then*

$$\lim_{n\to\infty} \frac{f_n(x)}{n} = \lim_{m\to\infty} \frac{1}{\lambda(X)} \int \frac{f_m}{m} d\lambda$$

*for almost every $x \in X$.*

*Remark* 5.2. If $f \in L^1(X, \lambda)$ and $f_n := \sum_{j=0}^{n-1} f(T^j x)$, then $f_{n+m} = f_n + f_m \circ T^n$ for all $n, m \in \mathbb{N}$. Further, since $\lambda$ is $T$-invariant,

$$\int \frac{f_m}{m} d\lambda = \frac{1}{m}\sum_{j=0}^{m-1} \int f(T^j x) d\lambda = \frac{1}{m}\sum_{j=0}^{m-1} \int f(x) dT_*^j\lambda = \int f d\lambda.$$

So the Kingman ergodic Theorem implies the Birkhoff ergodic theorem.

Before proving the Kingman ergodic theorem, we will apply it to random products. For the rest of the section let $G$, $\mu$, $\Omega = G^{\mathbb{N}}$, $\beta = \mu^{\otimes\mathbb{N}}$, etc. be as usual.

5.1. **Groups acting by isometries.** Suppose $G$ acts by isometries on a metric space $(X, \mathrm{d})$, that is

$$\mathrm{d}_X(gx_1, gx_2) = \mathrm{d}_X(x_1, x_2)$$

for all $g \in G$ and $x_1, x_2 \in X$.

**Theorem 5.3.** *If $x_0 \in X$ and*

$$\int_G \mathrm{d}_X(x_0, gx_0) d\mu(g) < +\infty,$$

*then there exists a number $L_\mu \ge 0$, called the **linear drift of** $\mu$, such that*

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{d}_X(x_0, w_n x_0) = L_\mu$$

*almost surely.*

*Remark* 5.4. If $x \in X$, then $|\mathrm{d}_X(x_0, gx_0) - \mathrm{d}_X(x, gx)| \le 2\,\mathrm{d}_X(x_0, x)$ and so the choice of $x_0$ is immaterial.

*Proof.* Define $f_n : \Omega \to \mathbb{R}$ by $f_n(\mathbf{g}) = \mathrm{d}_X(x_0, g_1 \cdots g_n x_0)$. Then

$$\begin{aligned} f_{n+m}(\mathbf{g}) &= \mathrm{d}_X(x_0, g_1 \cdots g_{n+m} x_0) \leq \mathrm{d}_X(x_0, g_1 \cdots g_n x_0) + \mathrm{d}(g_1 \cdots g_n x_0, g_1 \cdots g_{n+m} x_0) \\ &= f_n(\mathbf{g}) + f_m(T^n \mathbf{g}). \end{aligned}$$

Hence $\{f_n\}$ is subadditive. So by Kingman's subadditive ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{d}_X(x_0, w_n x_0) = \lim_{n\to\infty} \frac{1}{n} f_n(\mathbf{g}) = L$$

almost surely where

$$L := \inf_{m\geq 1} \int_\Omega \frac{1}{m} f_m(\mathbf{g}) d\beta(\mathbf{g}) = \inf_{m\geq 1} \int_G \frac{1}{m} \mathrm{d}_X(x_0, gx_0) d\mu^{*m}(g).$$

□

5.2. **Groups of matrices.** Suppose $G = \mathsf{SL}(d, \mathbb{R})$ and let

$$\|g\| = \max\{\|gv\| : v \in \mathbb{S}^{d-1}\}.$$

By the singular value decomposition $\|g\| = \sigma_1(g)$ and by definition

$$\|gh\| \leq \|g\| \, \|h\| .$$

**Theorem 5.5.** *Suppose*

$$\int_G \log \|g\| \, d\mu(g) < +\infty.$$

*The number*

$$\lambda_p(\mu) := \lim_{m\to\infty} \frac{1}{m} \int_G \log \sigma_p(g) d\mu^{*m}(g)$$

*is called the* $p^{th}$***-Lyapunov exponent of*** $\mu$ *and satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \log \sigma_p(g_1 \cdots g_n) = \lambda_p(\mu) = \lim_{n\to\infty} \frac{1}{n} \log \sigma_p(g_n \cdots g_1)$$

*almost surely.*

*Remark* 5.6. Notice that $\lambda_1(\mu) \geq \cdots \geq \lambda_d(\mu)$.

*Proof.* We first handle the $p = 1$ case.

**Lemma 5.7.** $\lambda_1(\mu) = \lim_{m\to\infty} \frac{1}{m} \int_G \log \|g\| \, d\mu^{*m}(g)$ *exists and*

$$\lim_{n\to\infty} \frac{1}{n} \log \|g_1 \cdots g_n\| = \lambda_1(\mu) = \lim_{n\to\infty} \frac{1}{n} \log \|g_n \cdots g_1\|$$

*almost surely.*

*Proof.* Define $f_n : \Omega \to \mathbb{R}$ by $f_n(\mathbf{g}) = \log \|g_1 \cdots g_n\|$. Then

$$\begin{aligned} f_{n+m}(\mathbf{g}) &= \log \|g_1 \cdots g_{n+m}\| \leq \log \Big( \|g_1 \cdots g_n\| \, \|g_{n+1} \cdots g_{n+m}\| \Big) \\ &= f_n(\mathbf{g}) + f_m(T^n \mathbf{g}). \end{aligned}$$

Hence $\{f_n\}$ is subadditive. Notice that if $g \in \mathsf{SL}(d, \mathbb{R})$, then $1 = \det(g) = \prod \sigma_k(g)$ and so $\|g\| = \sigma_1(g) \geq 1$. Hence $f_n \geq 0$. Then, by subadditivity

$$\int_\Omega |f_n| \, d\beta \leq \int_\Omega \sum_{j=0}^{n-1} \left| f_1 \circ T^j \right| d\beta = \int_G \sum_{j=0}^{n-1} |f_1| \, d\mu = n \int_G \log \|g\| \, d\mu(g) < +\infty.$$

So $\{f_n\} \subset L^1(\Omega, \beta)$.

So by Kingman's subadditive ergodic theorem,

$$\lim_{n\to\infty}\frac{1}{n}\log\|g_1\cdots g_n\| = \lim_{n\to\infty}\frac{1}{n}f_n(\mathbf{g}) = \lambda_1$$

almost surely where

$$\lambda_1 = \lim_{m\to\infty}\frac{1}{m}\int_\Omega f_m(\mathbf{g})d\beta(\mathbf{g}) = \lim_{m\to\infty}\frac{1}{m}\int_G \log\|g\|\, d\mu^{*m}(g).$$

Next define $h_n : \Omega \to \mathbb{R}$ by $h_n(\mathbf{g}) = \log\|g_n\cdots g_1\|$. By the same reasoning

$$\lim_{n\to\infty}\frac{1}{n}\log\|g_n\cdots g_1\| = \lambda_1$$

almost surely, since

$$\lim_{m\to\infty}\frac{1}{m}\int_\Omega h_m(\mathbf{g})d\beta(\mathbf{g}) = \lim_{m\to\infty}\frac{1}{m}\int_G \log\|g\|\, d\mu^{*m}(g) = \lambda_1. \qquad \square$$

To obtain the $p > 1$ case we consider $\wedge^p\mathbb{R}^d$, the $p^{th}$ exterior power of $\mathbb{R}^d$. Let $\tau_p : \mathsf{SL}(d,\mathbb{R}) \to \mathsf{SL}(\wedge^p\mathbb{R})$ be the standard representation given by

$$\tau_p(g)(v_1\wedge\cdots\wedge v_p) = (gv_1)\wedge\cdots\wedge(gv_p).$$

If we identify $\wedge^p\mathbb{R}^d \cong \mathbb{R}^{\dim\wedge^p\mathbb{R}^d}$ by mapping $\{e_{i_1}\wedge\cdots\wedge e_{i_p} : i_1 < \cdots < i_p\}$ to the standard basis, then the singular values of $\tau_p(g)$ are

$$\{\sigma_{i_1}(g)\cdots\sigma_{i_p}(g) : 1 \le i_1 < \cdots < i_p \le d\}.$$

Hence

$$\sigma_1(\tau_p(g)) = \sigma_1(g)\cdots\sigma_p(g).$$

Then, applying Lemma 5.7 to the measure $(\tau_p)_*\mu$ on $\mathsf{SL}(\wedge^p\mathbb{R}^d)$,

$$\lim_{n\to\infty}\frac{1}{n}\log(\sigma_1\cdots\sigma_p)(g_1\cdots g_n) = L_p = \lim_{n\to\infty}\frac{1}{n}\log(\sigma_1\cdots\sigma_p)(g_n\cdots g_1)$$

almost surely where

$$L_p := \lim_{m\to\infty}\frac{1}{m}\int_G \log(\sigma_1\cdots\sigma_p)(g)d\mu^{*m}(g).$$

Then

$$\lim_{n\to\infty}\frac{1}{n}\log\sigma_p(g_1\cdots g_n) = L_p - L_{p-1} = \lim_{n\to\infty}\frac{1}{n}\log\sigma_p(g_n\cdots g_1)$$

almost surely and $L_p - L_{p-1} = \lambda_p(\mu)$.

$\square$

## 6. Proof of the Kingman Ergodic Theorem

The proof we provide is adapted from [Lal].

We first prove a limiting result for subadditive sequences.

**Lemma 6.1.** *If $\{a_n\} \subset \mathbb{R}$ is subadditive, i.e. $a_{n+m} \le a_n + a_m$ for all $n, m \in \mathbb{N}$, then*

$$\lim_{n\to\infty}\frac{a_n}{n} = \inf_{m\ge1}\frac{a_m}{m}.$$

*Proof.* Fix $m \in \mathbb{N}$. Then any $n \in \mathbb{N}$ can be written as $n = p_n m + r_n$ where $p_n \in \mathbb{Z}_{\geq 0}$ and $r_n \in \{0, \ldots, m-1\}$. Then

$$\frac{a_n}{n} \leq \frac{p_n a_m}{n} + \frac{a_{r_n}}{n}.$$

Since $\frac{p_n}{n} \to \frac{1}{m}$, we then have

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_m}{m}.$$

Since $m \in \mathbb{N}$ was arbitrary,

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \inf_{m\geq 1} \frac{a_m}{m} \leq \liminf_{n\to\infty} \frac{a_n}{n}. \qquad \square$$

In the context of Theorem 5.1, since $T_*\lambda = \lambda$, the sequence $\{\int f_n d\lambda\}$ is subadditive and hence

$$\lim_{n\to\infty} \int \frac{f_n}{n} d\lambda = \inf_{m\geq 1} \int \frac{f_m}{m} d\lambda.$$

**Lemma 6.2.** *In Theorem 5.1, it suffices to consider the case when $f_n \leq 0$.*

*Proof.* Given a general $\{f_n\}$, let

$$\tilde{f}_n := f_n - \sum_{j=0}^{n-1} f_1 \circ T^j.$$

Since $\{f_n\}$ is subadditive, $\tilde{f}_n \leq 0$. Further,

$$\tilde{f}_n + \tilde{f}_m \circ T^n = f_n + f_m \circ T^n - \sum_{j=0}^{n+m-1} f_1 \circ T^j \geq f_{n+m} - \sum_{j=0}^{n+m-1} f_1 \circ T^j = \tilde{f}_{n+m}.$$

So $\{\tilde{f}_n\}$ is subadditive.

Assume that Theorem 5.1 is true for $\{\tilde{f}_n\}$. Then by the Birkhoff ergodic theorem

$$\lim_{n\to\infty} \frac{f_n(x)}{n} = \lim_{n\to\infty} \frac{\tilde{f}_n(x)}{n} + \frac{1}{n} \sum_{j=0}^{n-1} f_1 \circ T^j(x) = \lim_{m\to\infty} \frac{1}{\lambda(X)} \int \frac{\tilde{f}_m}{m} + f_1 d\lambda$$

for almost every $x \in X$. Since $\lambda$ is $T$-invariant,

$$\int \frac{\tilde{f}_m}{m} + f_1 d\lambda = \int \frac{f_m}{m} d\lambda$$

and so Theorem 5.1 is true for $\{f_n\}$ $\square$

The rest of the section is devoted to the proof of Theorem 5.1. So fix a non-positive subadditive sequence $\{f_n\}$. We can also scale $\lambda$ so that $\lambda(X) = 1$.

Let

$$c := \inf_{m\geq 1} \int \frac{f_m}{m} d\lambda = \lim_{m\to\infty} \int \frac{f_m}{m} d\lambda.$$

We will prove that $\limsup_{n\to\infty} \frac{f_n(x)}{n} \leq c$ and $\liminf_{n\to\infty} \frac{f_n(x)}{n} \geq c$ almost everywhere.

The non-positivity of each $f_n$ allows us to extend the subadditivity property as follows.

**Lemma 6.3.** *If $0 \leq j_1 < j_2 < \cdots < j_m \leq n$ and $1 \leq k_i \leq j_{i+1} - j_i$, then*

$$f_n \leq f_{k_1} \circ T^{j_1} + \cdots + f_{k_m} \circ T^{j_m}.$$

*Proof.* Note

$$\begin{aligned} f_n &\le f_{j_m} + f_{n-j_m} \circ T^{j_m} \le f_{j_m} + (f_{k_m} + f_{n-j_m-k_m} \circ T^{k_m}) \circ T^{j_m} \\ &\le f_{j_m} + f_{k_m} \circ T^{j_m}. \end{aligned}$$

Now induct. □

**Lemma 6.4.** $\limsup_{n\to\infty} \frac{f_n(x)}{n} \le c$ *for almost every* $x \in X$.

*Proof.* Fix $m \in \mathbb{N}$. Then any $n \in \mathbb{N}$ can be written as $n = p_n m + r_n$ where $p_n = \lfloor \frac{n}{m} \rfloor$ and $r_n \in \{0, 1, \dots, m-1\}$. Apply Lemma 6.3 to $0 \le m < 2m < \cdots < p_n m \le n$ to obtain

$$f_n(x) \le \sum_{j=0}^{p_n} f_m(T^{jm}x).$$

Further, for $1 \le \ell \le m-1$, we can apply Lemma 6.3 to $0 \le \ell < m+\ell < 2m+\ell < \cdots < (p_n-1)m + \ell \le n$ to obtain

$$f_n(x) \le \sum_{j=0}^{p_n-1} f_m(T^{jm+\ell}x).$$

Averaging these estimates we have

$$f_n(x) \le \frac{1}{m} \sum_{j=0}^{p_n m} f_m(T^j x).$$

By the Birkhoff ergodic theorem

$$\lim_{p\to\infty} \frac{1}{pm+1} \sum_{j=0}^{pm} f_m(T^j x) = \int f_m d\lambda$$

for almost every $x \in X$. Since $\frac{p_n m+1}{n} \to 1$,

$$\limsup_{n\to\infty} \frac{f_n(x)}{n} \le \int \frac{f_m}{m} d\lambda$$

for almost every $x \in X$. Since $m \in \mathbb{N}$ is arbitrary, the lemma follows. □

Let

$$F(x) = \liminf_{n\to\infty} \frac{f_n(x)}{n} \in [-\infty, 0].$$

**Lemma 6.5.** *$F$ is constant almost everywhere.*

*Proof.* By subadditivity and non-positivity,

$$f_n \le f_1 + f_{n-1} \circ T \le f_{n-1} \circ T.$$

Hence $F \le F \circ T$. In particular, if $E_c = \{F(x) \le c\}$, then $T^{-1}E_c \subset E_c$ (if $x \in T^{-1}E_c$, then $F(Tx) \le c$ and so $F(x) \le c$). However, since $\lambda$ is $T$-invariant, $\lambda(T^{-1}E_c) = \lambda(E_c)$. So $\lambda(E_c \Delta T^{-1}E_c) = 0$ and $E_c$ has either zero or full measure. Then

$$F = \inf\{c : E_c \text{ has full measure}\}$$

almost everywhere. □

**Lemma 6.6.** $F(x) = \liminf_{n\to\infty} \frac{f_n(x)}{n} \ge c$ *for almost every* $x \in X$.

*Proof.* Suppose not. Since $F$ is constant almost everywhere, then there exists $\epsilon > 0$ such that the set $\{x : F(x) < c - \epsilon\}$ has full measure. Fix $\delta > 0$. Since

$$F(x) = \liminf_{n\to\infty} \frac{f_n(x)}{n} \geq \inf_{n\geq 1} \frac{f_n(x)}{n},$$

there exists $N_0$ such that the set

$$B := \left\{ x : \min_{1\leq k\leq N_0} \frac{f_k(x)}{k} < c - \epsilon \right\}$$

satisfies $\lambda(B) > \lambda(X) - \delta = 1 - \delta$.

**Claim:** $f_{nN_0+N_0}(x) < (c-\epsilon)\sum_{j=1}^{nN_0} 1_B(T^j x)$.

Fix $x \in X$. Let $J := \{j : 1 \leq j \leq nN_0 \text{ and } T^j x \in B\}$. Let $j_1 \in J$ be the smallest element in $J$. Then there exists $1 \leq k_1 \leq N_0$ such that

$$f_{k_1}(T^{j_1}x) < (c-\epsilon)k_1.$$

Then let $j_2 \in J$ be the smallest element in $J$ greater or equal to than $j_1 + k_1$. Then there exists $1 \leq k_2 \leq N_0$ such that

$$f_{k_2}(T^{j_2}x) < (c-\epsilon)k_2.$$

Repeating this process we find $1 \leq j_1 < \cdots < j_r \leq nN_0$ and $k_1, \ldots, k_r$ satisfying $k_i \leq j_{i+1} - j_i$ and

$$f_{k_i}(T^{j_i}x) < (c-\epsilon)k_i.$$

By construction

$$J \subset \bigcup_{i=1}^{r} [j_i, j_i + k_i - 1]$$

and so $|J| \leq \sum k_i$. Thus by Lemma 6.3

$$f_{nN_0+N_0}(x) \leq f_{k_1}(T^{j_1}x) + \cdots + f_{k_r}(T^{j_r}x) < (c-\epsilon)\sum_{i=1}^{r} k_i \leq (c-\epsilon)\,|J|$$

(recall that $c - \epsilon < 0$). Thus the claim is proven.

Now if we integrate both sides of the claim we obtain

$$\frac{1}{N_0 n}\int f_{nN_0+N_0}(x)d\lambda < (c-\epsilon)\lambda(B) < (c-\epsilon)(1-\delta).$$

Then

$$c = \lim_{n\to\infty} \frac{1}{nN_0}\int f_{nN_0+N_0}(x)d\lambda \leq (c-\epsilon)(1-\delta).$$

Since $\delta > 0$ was arbitrary, we have a contradiction. □

## 7. Stationary measures

Let $G$, $\mu$, $\Omega = G^{\mathbb{N}}$, $\beta = \mu^{\otimes\mathbb{N}}$, etc. be as usual. In this section we consider the situation where $G$ acts continuously on a compact metrizable space $Y$, i.e. the map

$$(g, y) \in G \times Y \mapsto gy \in Y$$

is continuous. We will introduce stationary measures on $Y$, which will play an important role in these lecture notes.

Given a probability measure $\nu$ on $Y$ we can obtain a new probability measure $\mu * \nu$, called the *convolution of $\mu$ and $\nu$*, by averaging using $\mu$:

$$\mu * \nu = \int_G g_* \nu d\mu(g).$$

Notice that for a measurable set $A \subset Y$,

$$\mu * \nu(A) = \int_G \nu(g^{-1}A) d\mu(g)$$

and for an integrable function

$$\int f d(\mu * \nu) = \int_G \int_Y f(gy) d\nu(y) d\mu(g).$$

*Remark* 7.1. $G$ acts on $G$ on the left and in this case, $\mu * \mu$ coincides with the definition above.

We also observe that the convolution is the push-forward of the product measure $\mu \otimes \nu$ on $G \times Y$ under the action of the group.

**Observation 7.2.** If $p : G \times Y \to Y$ is the map $p(g, x) = g \cdot x$, then $\mu * \nu = p_*(\mu \otimes \nu)$.

*Proof.* If $A \subset Y$ is Borel measurable, then

$$p^{-1}(A) = \{(g, y) : y \in g^{-1}A\}.$$

So

$$p_*(\mu \otimes \nu)(A) = \mu \otimes \nu(p^{-1}(A)) = \int_G \nu(g^{-1}A) d\mu(g) = \mu * \nu(A).$$

□

We are interested in measures that are invariant under convolution with $\mu$.

**Definition 7.3.** A probability measure $\nu$ on $Y$ is called *stationary* if $\mu * \nu = \nu$.

We can think about $\mu$ inducing a "random group action on $Y$": one acts by an element $g$ with probability given by $\mu$. Then a stationary measure is a measure which is invariant under this "random group action."

The next result shows that stationary measures always exist.

**Observation 7.4.** Suppose $y_0 \in Y$ and $\delta_{y_0}$ is the Dirac mass centered at $y_0$. Let

$$\nu_n := \frac{1}{n} \sum_{k=0}^{n-1} \mu^{*k} * \delta_{y_0}.$$

Then any weak-$*$ limit point of $\{\nu_n\}$ is a stationary measure.

*Proof.* Suppose $\nu_{n_j} \to \nu$ is the weak-$*$ topology. Then, $\mu * \nu_{n_j} \to \mu * \nu$ in the weak-$*$ topology (check!). Hence

$$\begin{aligned}
\mu * \nu &= \lim_{j \to \infty} \mu * \nu_{n_j} = \lim_{j \to \infty} \frac{1}{n_j} \mu * \left( \sum_{k=0}^{n_j - 1} \mu^{*k} * \delta_{y_0} \right) \\
&= \lim_{j \to \infty} \frac{1}{n_j} \sum_{k=1}^{n_j} \mu^{*k} * \delta_{y_0} = \lim_{j \to \infty} \nu_{n_j} - \frac{1}{n_j} \delta_{y_0} + \frac{1}{n_j} \mu^{*n_j} * \delta_{y_0} \\
&= \nu - 0 + 0.
\end{aligned}$$

So $\mu * \nu = \nu$.

□

Observation 7.4 says that stationary measures always exist and when there is a unique stationary measure the Observation gives us the following.

**Observation 7.5.** If there is a unique stationary measure $\nu$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\Pr(w_k\cdot y_0\in A)=\nu(A)$$

for any $y_0\in Y$ and Borel subset $A\subset Y$.

*Proof.* Let $\nu_n$ be as in Observation 7.4. Since $\nu$ is the only stationary measure, we must have $\nu_n\to\nu$. Further,

$$(\mu^{*k}*\delta_{y_0})(A)=\int_G\delta_{gy_0}(A)d\mu^{*k}(g)=\int_\Omega\delta_{w_ky_0}(A)d\beta=\Pr(w_k\cdot y_0\in A)$$

and so the result follows. □

## 8. Ergodic stationary measures

Let $G$, $\mu$, $\Omega=G^{\mathbb{N}}$, $\beta=\mu^{\otimes\mathbb{N}}$, etc. be as usual. Also, assume that $G$ acts continuously on a compact metrizable space $Y$. In this section we introduce and study ergodic stationary measures.

**Definition 8.1.** A stationary measure $\nu$ is *ergodic* if it is an extreme point of the set of stationary measures, i.e. if $\nu=t\nu_1+(1-t)\nu_2$ where $0<t<1$ and $\nu_1,\nu_2$ are stationary measures, then $\nu=\nu_1=\nu_2$.

*Remark* 8.2. Let $\mathcal{E}_\mu(Y)$ denote the space of ergodic stationary measures. It follows from a theorem in functional analysis (Choquet's theorem), that for any stationary measure $\nu_0$ there exists a probability measure $\lambda$ on $\mathcal{E}_\mu(Y)$ such that

$$\nu_0=\int_{\mathcal{E}_\mu(Y)}\nu d\lambda(\nu)$$

(informally $\nu_0$ is a convex combination of elements in $\mathcal{E}_\mu(Y)$). So ergodic stationary measures always exist and one can often reduce to the case of an ergodic stationary measure.

In this section we will establish several characterizations of ergodicity.

8.1. **An operator:** Given a stationary measure $\nu$ define $P_\mu:L^1(Y,\nu)\to L^1(Y,\nu)$ by

$$P_\mu(f)(y)=\int_G f(gy)d\mu(g).$$

**Observation 8.3.** $P_\mu$ is well-defined on $L^1(Y,\nu)$ and $\|P_\mu\|=1$.

*Proof.* If $f\in L^1(Y,\nu)$, then

$$\|f\|_{L^1}=\int_Y|f|\,d\nu=\int_Y|f|\,d(\mu*\nu)=\int_Y\int_G|f(gy)|\,d\mu(g)d\nu(y)$$

and hence $\int_G|f(gy)|\,d\mu(g)<+\infty$ for $\nu$-almost every $y$. So $P_\mu(f)(y)$ is defined $\nu$-almost everywhere. Also

$$\|P_\mu f\|_{L^1}=\int_Y|P_\mu(f)|\,d\nu=\int_Y\left|\int_G f(gy)d\mu(g)\right|d\nu(y)\le\int_Y\int_G|f(gy)|\,d\mu(g)d\nu(y)=\|f\|_{L^1}.$$

Thus $P_\mu : L^1(Y,\nu) \to L^1(Y,\mu)$ and also $\|P_\mu\| \leq 1$. The above calculation also shows that $\|P_\mu f\|_{L^1} = \|f\|_{L^1}$ when $f \geq 0$ and so $\|P_\mu\| = 1$.

□

8.2. **A dynamical system.** Recall that $T : \Omega \to \Omega$ is defined by

$$T(g_1, g_2, g_3, \dots) = (g_2, g_3, \dots).$$

Define $S : \Omega \times Y \to \Omega \times Y$ by

$$S(\mathbf{g}, y) = (T(\mathbf{g}), g_1 y) = \big((g_2, g_3, \dots), g_1 y\big).$$

**Observation 8.4.** If $\nu$ is a stationary measure, then the measure $\beta \otimes \nu$ is $S$-invariant.

*Proof.* It suffices to consider sets of the form $A \times B$. Notice that

$$S^{-1}(A \times B) = \big\{(\mathbf{g}, y) : \mathbf{g} \in T^{-1}(A),\ y \in g_1^{-1} B\big\}$$

and so

$$\begin{aligned}\beta \otimes \nu(S^{-1}(A \times B)) &= \int_{T^{-1}(A)} \nu(g_1^{-1}B) d\beta = \int_A \int_G \nu(g_1^{-1}B) d\mu(g_1) d\beta \\ &= \int_A (\mu * \nu)(B) d\beta = \int_A \nu(B) d\beta = \beta \otimes \nu(A \times B).\end{aligned}$$

□

8.3. **Characterizations of Ergodicity.** The following proof comes from [Fur02, Theorem 3.1], which itself comes from a book by Kifer [Kif86, p. 19-20]. It is a special case of what is sometimes called the "Random Ergodic Theorem."

**Proposition 8.5.** *If $\nu$ is a stationary measure, then the following are equivalent:*

1. *$\nu$ is ergodic.*
2. *$S$ is ergodic with respect to $\beta \otimes \nu$.*
3. *Every measurable subset $A \subset Y$ for which $\nu(A \Delta g^{-1} A) = 0$ for $\mu$-almost every $g \in G$ either has full or zero $\nu$-measure.*
4. *Every $f \in L^1(X,\nu)$ for which $P_\mu f = f$ $\nu$-almost everywhere is constant $\nu$-almost everywhere.*

*Proof.* (2) ⇒ (1). We prove the contrapositive: if $\nu$ is not ergodic, then $S$ is not ergodic with respect to $\beta^Y$. Since $\nu$ is not ergodic, $\nu = t\nu_1 + (1-t)\nu_2$ where $0 < t < 1$ and $\nu_1, \nu_2$ are distinct stationary measures. Then $\beta \otimes \nu = t\beta \otimes \nu_1 + (1-t)\beta \otimes \nu_2$ and each $\beta \otimes \nu_i$ is $S$-invariant by Observation 8.4. So $S$ is not ergodic with respect to $\beta \otimes \nu$, see Proposition 2.6.

(1) ⇒ (3). Suppose for a contradiction that $\nu(A \Delta g^{-1} A) = 0$ for $\mu$-almost every $g \in G$ and $\nu(A), \nu(A^c)$ are both positive. Then $\nu_1 := \frac{1}{\nu(A)} \nu(A \cap \cdot)$ and $\nu_2 := \frac{1}{\nu(A^c)} \nu(A^c \cap \cdot)$ are both stationary measures (check!) and

$$\nu = \nu(A)\nu_1 + \nu(A^c)\nu_2 = \nu(A)\nu_1 + (1 - \nu(A))\nu_2.$$

So we have a contradiction.

(3) ⇒ (4). Suppose $f \in L^1(X,\nu)$ and $P_\mu f = f$ $\nu$-almost everywhere. Then

$$|f| = |P_\mu f| \leq P_\mu |f|$$

$\nu$-almost everywhere. So

$$\int_Y |P_\mu |f| - |f|| \, d\nu = \int_Y P_\mu |f| - |f| \, d\nu = 0$$

since

$$\int_Y P_\mu |f| \, d\nu = \int_Y \int_G |f| \circ g d\mu(g) d\nu = \int_Y |f| \, d(\mu * \nu) = \int_Y |f| \, d\nu.$$

Hence $P_\mu |f| = |f|$ $\nu$-almost everywhere.

Let $f^+ := \frac{f+|f|}{2} = \max\{f, 0\}$. Then $P_\mu f^+ = f^+$ $\nu$-almost everywhere. Let $E := \{y : f(y) \geq 0\}$. Then if $y \in E$, we have

$$P_\mu f^+(y) = f^+(y) = f(y) = P_\mu f(y)$$

which implies that $\nu(E \Delta g^{-1} E) = 0$ for $\mu$-almost every $g \in G$. Thus $E$ either has full or zero $\nu$-measure.

Applying the same argument to $f - c$ we see that for any $c \in \mathbb{R}$ the set $E_c := \{y : f(y) \geq c\}$ either has full or zero $\nu$-measure. Hence

$$f = \sup\{c : \nu(E_c) = 1\}$$

$\nu$-almost everywhere.

(4) $\Rightarrow$ (2). Suppose $E \subset \Omega \times Y$ is $S$-invariant. Then $1_E \circ S = 1_E$. Since $\beta \otimes \nu$ is a probability measure, then $1_E \in L^2 = L^2(\Omega \times Y, \beta^Y)$.

For $n = 0, 1, \ldots$ let $\mathcal{H}_n \subset L^2$ be the subspace of functions which only depend on $g_1, \ldots, g_n$ and $y$. Then each $\mathcal{H}_n$ is a closed linear subspace and $\cup \mathcal{H}_n$ is dense in $L^2$. Further, the orthogonal projection $P_n : L^2 \to \mathcal{H}_n$ is given by

$$P_n(f)(\mathbf{g}, y) = \int_\Omega f(g_1, \ldots, g_n, \mathbf{h}, y) d\beta(\mathbf{h}).$$

Let $f_n := P_n(1_E)$, which we view as a function on $G^n \times Y$. Since $\cup \mathcal{H}_n$ is dense, $f_n \to 1_E$ in $L^2$. Further, since $1_E \circ S = 1_E$, we have

$$\begin{aligned} f_n(g_1, \ldots, g_n, y) &= \int_\Omega 1_E(g_1, \ldots, g_n, \mathbf{h}, y) d\beta(\mathbf{h}) = \int_\Omega 1_E(g_2, \ldots, g_n, \mathbf{h}, g_1 y) d\beta(\mathbf{h}) \\ &= f_{n-1}(g_2, \ldots, g_n, g_1 y). \end{aligned}$$

Repeating we see that $f_n(g_1, \ldots, g_n, y) = f_0(g_n \cdots g_1 y)$.

Further,

$$\begin{aligned} f_0(y) &= \int_\Omega 1_E(\mathbf{h}, y) d\beta(\mathbf{h}) = \int_\Omega \int_G 1_E(h_1, \mathbf{h}', y) d\mu(h_1) d\beta(\mathbf{h}') = \int_G f_1(h_1, y) d\mu(h_1) \\ &= \int_G f_0(h_1 y) d\mu(h_1) = P_\mu f_0(y). \end{aligned}$$

So by assumption $f_0$ is constant $\nu$-almost everywhere (note $f_0$ is bounded and hence in $L^1(Y, \nu)$). Thus each $f_n$ is constant $\mu^{\otimes n} \otimes \nu$-almost everywhere. Since $f_n \to 1_E$ in $L^2$, then we see that $1_E$ is constant $\beta \otimes \nu$-almost everywhere. Hence $E$ has either zero or full measure. Thus $S$ is ergodic with respect to $\beta \otimes \nu$.

□

## 9. Limit measures

In this section we prove that a stationary measure disintegrates into a collection of "limit measures" obtained by applying the random walk to the stationary measure. Our exposition follows [BQ16, Section 2.5]. Similar arguments appear in [Lal23, Lemma 11.4.8].

Let $G$, $\mu$, $\Omega = G^{\mathbb{N}}$, $\beta = \mu^{\otimes \mathbb{N}}$, etc. be as usual. Also, assume that $G$ acts continuously on a compact metrizable space $Y$.

Let $\mathcal{P}(Y)$ denote the space of Borel probability measures on $Y$ endowed with the weak-$*$ topology.

**Theorem 9.1.** *If $\nu$ is a stationary measure, then there exists a Borel map $\mathbf{g} \in \Omega \mapsto \nu_{\mathbf{g}} \in \mathcal{P}(Y)$ such that for $\beta$-almost every $\mathbf{g} = (g_1, g_2, \dots)$ one has*

$$(g_1 \cdots g_n)_* \nu \to \nu_{\mathbf{g}}.$$

*Moreover,*

$$\nu = \int \nu_{\mathbf{g}} d\beta(\mathbf{g}).$$

To prove the theorem we need to recall a result from probability theory.

9.1. **Doob's martingale convergence theorem.** We start by recalling the definition of a martingale.

**Definition 9.2.** Suppose $(X, \mathcal{A}, \lambda)$ is a probability space.

- A *filtration* of $(X, \mathcal{A}, \lambda)$ is an increasing sequence of sigma-subalgebras $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$ in $\mathcal{A}$.
- A sequence $\{f_n\}$ in $L^1(X, \mathcal{A}, \lambda)$ is *adapted* to a filtration $\{\mathcal{A}_n\}$ if each $f_n$ is $\mathcal{A}_n$-measurable, i.e. $f_n^{-1}(A) \in \mathcal{A}_n$ for any Borel subset $A \subset \mathbb{R}$.
- An adapted sequence $\{f_n\}$ is a *martingale (with respect to $\{\mathcal{A}_n\}$)* if

$$\int_A f_{n+m} d\lambda = \int_A f_n d\lambda$$

for all $A \in \mathcal{A}_n$ and $m, n \geq 0$.

**Example 9.3.** Consider $[0,1]^d$ with the Lebesgue measure. Let $\mathcal{A}_n$ be the sigma algebra generated by cubes of the form

$$\prod_{i=1}^{d} \left[ \frac{p_i}{2^n}, \frac{p_i + 1}{2^n} \right].$$

Then $\{\mathcal{A}_n\}$ is a filtration. Given $f \in L^1$ we can define a martingale $\{f_n\}$ where $f_n$ is constant on each cube of the form $\prod \left[ \frac{p_i}{2^n}, \frac{p_i+1}{2^n} \right]$ and equals the average value of $f$ on this cube. Further, by the Lebesgue differentiation theorem $f_n$ converges to $f$ almost everywhere.

**Theorem 9.4** (Doob's martingale convergence theorem)**.** *Suppose $(X, \mathcal{A}, \lambda)$ is a probability space, $\{\mathcal{A}_n\}$ is a filtration, and $\{f_n\}$ is a martingale. If $\sup_{n \geq 0} \|f_n\|_{L^1} < +\infty$, then there exists $f \in L^1$ such that $f_n(x) \to f(x)$ almost everywhere.*

We will only use the theorem when we have the stronger assumption that $\sup_{n \geq 0} \|f_n\|_{L^\infty} < +\infty$ and so we will just prove the theorem in the special case when $\sup_{n \geq 0} \|f_n\|_{L^2} < +\infty$ (following the proof in [BQ16, Appendix A.2]).

**Lemma 9.5.** *If $\{f_n\}$ is a martingale and $\epsilon > 0$,*

$$\lambda \left( \left\{ \sup_{1 \leq k \leq n} |f_k| \geq \epsilon \right\} \right) \leq \frac{1}{\epsilon} \int |f_n| \, d\lambda.$$

*Proof.* Note that $A := \{\sup_{1 \leq k \leq n} |f_k| \geq \epsilon\} = \sqcup_{k=1}^n A_k$ where

$$A_k := \{|f_1| < \epsilon, \dots, |f_{k-1}| < \epsilon, |f_k| \geq \epsilon\}.$$

Notice that $A_k \in \mathcal{A}_k$ (since each $f_n$ is $\mathcal{A}_n$-measurable). Further, $A_k^- := A_k \cap \{f_k < 0\}$ and $A_k^+ := A_k \cap \{f_k > 0\}$ are also in $\mathcal{A}_k$ and so when $n \geq k$,

$$\int_{A_k} |f_k| \, d\lambda = \int_{A_k^+} f_k d\lambda - \int_{A_k^-} f_k d\lambda = \int_{A_k^+} f_n d\lambda - \int_{A_k^-} f_n d\lambda \leq \int_{A_k} |f_n| \, d\lambda.$$

So

$$\lambda(A) = \sum_{k=1}^{n} \lambda(A_k) \leq \frac{1}{\epsilon} \sum_{k=1}^{n} \int_{A_k} |f_k| \, d\lambda \leq \frac{1}{\epsilon} \sum_{k=1}^{n} \int_{A_k} |f_n| \, d\lambda \leq \frac{1}{\epsilon} \int |f_n| \, d\lambda. \qquad \square$$

*Proof of Theorem 9.4 in the $L^2$ case.* Since each $f_m$ can be approximated by finite sums of the form $\sum c_i 1_{A_i}$ with $A_i \in \mathcal{A}_m$, we have

$$\int f_n f_m d\lambda = \int f_m^2 d\lambda$$

when $n \geq m$. Thus when $n \geq m$

$$0 \leq \int (f_n - f_m)^2 d\lambda = \int f_n^2 d\lambda - \int f_m^2 d\lambda = \|f_n\|_{L^2} - \|f_m\|_{L^2}.$$

So the sequence $\{\|f_n\|_{L^2}\}$ is non-decreasing and hence converges (being bounded). Then the above formula implies that $\{f_n\}$ is a Cauchy sequence in $L^2$. Hence $f_n \to f$ in $L^2$. Since $\lambda(X) = 1$, we also have $f_n \to f$ in $L^1$. Notice that for each $m \geq 1$ the sequence $\{f_n - f_m\}_{n \geq m}$ is a martingale, so using the lemma and taking a limit

$$\lambda\left(\left\{\sup_{n \geq m} |f_n - f_m| \geq \epsilon\right\}\right) \leq \frac{1}{\epsilon} \int |f - f_m| \, d\lambda$$

for all $m \geq 1$. Thus $f_n(x) \to f(x)$ on the full measure set

$$\bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \left\{\sup_{n \geq m} |f_n - f_m| \leq 1/k\right\}. \qquad \square$$

9.2. **Proof of Theorem 9.1.** We are now ready to prove Theorem 9.1, which we restate here.

**Theorem.** *With the usual notation, if $\nu$ is a stationary measure, then there exists a Borel map $\mathbf{g} \in \Omega \mapsto \nu_{\mathbf{g}} \in \mathcal{P}(Y)$ such that for $\beta$-almost every $\mathbf{g} = (g_1, g_2, \dots)$ one has*

$$(w_n)_* \nu = (g_1 \cdots g_n)_* \nu \to \nu_{\mathbf{g}}.$$

*Moreover,*

$$\nu = \int \nu_{\mathbf{g}} d\beta(\mathbf{g}).$$

*Proof.* Let $\mathcal{F}_n$ be the sigma-algebra generated by cylinders of the form

$$A_1 \times \cdots \times A_n \times G^{\mathbb{N}}.$$

Then $\{\mathcal{F}_n\}$ is a filtration and $f : \Omega \to \mathbb{R}$ is $\mathcal{F}_n$-measurable if and only if $f$ only depends on $g_1, \dots, g_n$.

Given $\varphi \in C(Y)$, define $\rho_n(\varphi) : \Omega \to \mathbb{R}$ by

$$\rho_n(\varphi)(\mathbf{g}) = \int_Y \varphi \, d(g_1 \cdots g_n)_* \nu = \int_Y \varphi(g_1 \cdots g_n y) d\nu(y).$$

Notice that $\rho_n(\varphi)$ is $\mathcal{F}_n$-measurable. Further, if $A = A_1 \times \cdots A_n \times G^{\mathbb{N}} \in \mathcal{F}_n$, then

$$\begin{aligned}\int_A \rho_{n+m}(\varphi)(\mathbf{g})d\beta(\mathbf{g}) &= \int_A \int_Y \varphi(g_1 \cdots g_{n+m}y)d\nu(y)d\beta(\mathbf{g}) \\ &= \int_A \int_Y \int_G \varphi(g_1 \cdots g_n \cdot hy)d\mu^{*m}(h)d\nu(y)d\beta(\mathbf{g}) \\ &= \int_A \int_Y \varphi(g_1 \cdots g_n y)d(\mu^{*m} * \nu)(y)d\beta(\mathbf{g}) \\ &= \int_A \int_Y \varphi(g_1 \cdots g_n y)d\nu(y)d\beta(\mathbf{g}) = \int_A \rho_n(\varphi)(\mathbf{g})d\beta(\mathbf{g}).\end{aligned}$$

Hence $\{\rho_n(\varphi)\}$ is a martingale. Since $\sup_{n\geq 1} \|\rho_n(\varphi)\|_1 < +\infty$, there exists $\rho(\varphi) \in L^1$ such that $\rho_n(\varphi) \to \rho(\varphi)$ almost everywhere.

Now fix a countable dense subset $D \subset C(Y)$. By the above, there exists a full measure subset $\Omega'$ such that $\rho_n(\varphi)(\mathbf{g}) \to \rho(\varphi)(\mathbf{g})$ for all $\varphi \in D$ and all $\mathbf{g} \in \Omega'$.

We claim that

$$\lim_{n\to\infty} (g_1 \cdots g_n)_*\nu$$

exists in $\mathcal{P}(Y)$ for all $\mathbf{g} \in \Omega'$. Fix $\mathbf{g} \in \Omega'$. If $\lambda$ is a limit point of $\{(g_1 \cdots g_n)_*\nu\}$, then

$$\int_Y \varphi \, d\lambda = \lim_{n\to\infty} \int_Y \varphi \, d(g_1 \cdots g_n)_*\nu = \lim_{n\to\infty} \rho_n(\varphi)(\mathbf{g}) = \rho(\varphi)(\mathbf{g})$$

for all $\varphi \in D$. Since $D \subset C(Y)$ is dense, there is only one limit point. Thus $\lim_{n\to\infty}(g_1 \cdots g_n)_*\nu$ exists.

Then

$$\nu_{\mathbf{g}} = \begin{cases} \lim_{n\to\infty}(g_1 \cdots g_n)_*\nu & \text{if } \mathbf{g} \in \Omega' \\ \nu & \text{if } \mathbf{g} \notin \Omega' \end{cases}$$

defines a Borel map $\Omega \to \mathcal{P}(Y)$ with the desired property.

Next we show that $\nu = \int \nu_{\mathbf{g}} d\beta(\mathbf{g})$. Fix $\varphi \in C(Y)$. Since $\nu$ is stationary,

$$\int_Y \varphi d\nu = \int_Y \int_\Omega \varphi(g_1 \cdots g_n y)d\beta(\mathbf{g})d\nu(y) = \int_\Omega \left[\int_Y \varphi d(g_1 \cdots g_n)_*\nu\right] d\beta(\mathbf{g}).$$

Then the dominated convergence theorem implies that

$$\int_Y \varphi d\nu = \int_\Omega \left[\int_Y \varphi d\nu_{\mathbf{g}}\right] d\beta(\mathbf{g}).$$

So $\nu = \int \nu_{\mathbf{g}} d\beta(\mathbf{g})$.

□

9.3. **A technical extension.** We will need the following technical extension of Theorem 9.1.

**Corollary 9.6.** *Suppose $\nu$ is a stationary measure and $\mathbf{g} \in \Omega \mapsto \nu_{\mathbf{g}} \in \mathcal{P}(Y)$ is the Borel map in Theorem 9.1. For every $m \in \mathbb{N}$, for $\mu^{*m} \otimes \beta$-almost every $(h, \mathbf{g})$ one has*

$$(g_1 \cdots g_n h)_*\nu \to \nu_{\mathbf{g}}.$$

*Proof.* It suffices to fix $\varphi \in C(Y)$ and show that for $\mu^{*m} \otimes \beta$-almost every $(h, \mathbf{g})$ one has

$$\int \varphi d(g_1 \cdots g_n h)_*\nu \to \int \varphi d\nu_{\mathbf{g}}.$$

Indeed, if the above is true, then fix a countable dense subset $D \subset C(Y)$. Then for for $\mu^{*m} \otimes \beta$-almost every $(h, \mathbf{g})$ one has

$$\int \varphi d(g_1 \cdots g_n h)_* \nu \to \int \varphi d\nu_{\mathbf{g}}$$

for all $\varphi \in D$. For such $(h, \mathbf{g})$ one then has that

$$\int \varphi d(g_1 \cdots g_n h)_* \nu \to \int \varphi d\nu_{\mathbf{g}}$$

for all $\varphi \in C(Y)$ by approximation.

So fix $\varphi \in C(Y)$. Define $f_n, f_n^h : \Omega \to \mathbb{R}$ by

$$f_n(\mathbf{g}) = \int \varphi d(g_1 \cdots g_n)_* \nu = \int \varphi(g_1 \cdots g_n y) d\nu(y)$$

and

$$f_n^h(\mathbf{g}) = \int \varphi d(g_1 \cdots g_n h)_* \nu = \int \varphi(g_1 \cdots g_n h y) d\nu(y).$$

By Theorem 9.1, $f_n(\mathbf{g}) \to \int \varphi d\nu_{\mathbf{g}}$ for $\beta$-almost every $\mathbf{g}$. So it suffices to show that $f_n(\mathbf{g}) - f_n^h(\mathbf{g}) \to 0$ for $\mu^{*m} \otimes \beta$-almost every $(h, \mathbf{g})$.

To simplify computations let $\Phi(g) = \int \varphi(gy) d\nu(y)$. Notice that since $\nu$ is stationary,

$$\int_G \Phi(gh) d\mu^{*m}(h) = \Phi(g).$$

Then, computing we have

$$\begin{aligned} I_n :&= \int_\Omega \int_G (f_n(\mathbf{g}) - f_n^h(\mathbf{g}))^2 d(\mu^{*m} \otimes \beta)(h, \mathbf{g}) \\ &= \int_\Omega \int_G \Phi(g_1 \cdots g_n)^2 - 2\Phi(g_1 \cdots g_n)\Phi(g_1 \cdots g_n h) + \Phi(g_1 \cdots g_n h)^2 d(\mu^{*m} \otimes \beta)(h, \mathbf{g}) \\ &= \int_\Omega \Phi(g_1 \cdots g_n)^2 - 2\Phi(g_1 \cdots g_n)^2 + \Phi(g_1 \cdots g_{n+m})^2 d\beta(\mathbf{g}) = J_{m+n} - J_n \end{aligned}$$

where $J_k := \int_G \Phi(g)^2 d\mu^{*k}(g)$. Since each $I_n$ is positive and

$$\sum_{n=1}^{N} I_n = \sum_{n=N+1}^{N+m} J_n - \sum_{n=1}^{m} J_n \leq 2m \left\|\varphi\right\|_\infty^2,$$

we see that $\sum_n I_n < +\infty$.

Thus for $\mu^{*m} \otimes \beta$-almost every $(h, \mathbf{g})$,

$$\sum_n (f_n(\mathbf{g}) - f_n^h(\mathbf{g}))^2 < +\infty,$$

which implies that $f_n(\mathbf{g}) - f_n^h(\mathbf{g}) \to 0$.

□

# Part 2. Linear groups

## 10. Statement of Results

Let $G = \mathsf{SL}(d, \mathbb{R})$ and let $\mu$ be a probability measure on $\mathsf{SL}(d, \mathbb{R})$. As usual, let $\Omega = G^{\mathbb{N}}$ and $\beta = \mu^{\otimes \mathbb{N}}$.

Recall that when $\int \log \|g\| \, d\mu(g) < +\infty$, the limits

$$\lambda_p(\mu) := \lim_{n\to\infty} \frac{1}{n} \log \sigma_p(g_1 \cdots g_n) = \lim_{n\to\infty} \frac{1}{n} \log \sigma_p(g_n \cdots g_1)$$

exist for $\beta$-almost every $\mathbf{g} \in \Omega$ and that the numbers

$$\lambda_1(\mu) \geq \cdots \geq \lambda_d(\mu)$$

are called the *Lyapunov exponents* of $\mu$.

Define $k_n, \ell_n : \Omega \to \mathsf{SO}(d)$ and $a_n : \Omega \to \{\text{diagonal matrices}\}$ so that

$$g_1 \cdots g_n = k_n(\mathbf{g}) a_n(\mathbf{g}) \ell_n(\mathbf{g})$$

is a singular value decomposition.

The goal of this part of the notes is to prove the following.

**Theorem 10.1** (Gol'dsheĭd–Margulis [GM89])**.** *If*

- $\int \log \|g\| \, d\mu(g) < +\infty$,
- *the group generated by* $\operatorname{supp}(\mu)$ *is Zariski dense in* $\mathsf{SL}(d, \mathbb{R})$,

*then*

$$\lambda_1(\mu) > \lambda_2(\mu) > \cdots > \lambda_d(\mu).$$

*Moreover, the limit*

$$\lim_{n\to\infty} k_n(\mathbf{g}) \, \mathsf{M}$$

*exists in* $\mathsf{SO}(d)/\mathsf{M}$ *for* $\beta$*-almost every* $\mathbf{g} \in \Omega$.

10.1. **Metric space interpretation.** Associated to $\mathsf{SL}(d, \mathbb{R})$ is a special metric space. Let $X = \mathsf{SL}(d, \mathbb{R})/\mathsf{K}$ where $\mathsf{K} = \mathsf{SO}(d)$ and define a distance on $X$ by

$$\mathrm{d}_X(g_1 \mathsf{K}, g_2 \mathsf{K}) = \sqrt{\sum_{j=1}^{d} [\log \sigma_j(g_1^{-1} g_2)]^2}.$$

Since $\sigma_j(k_1 g k_2) = \sigma_j(g)$ for all $k_1, k_2 \in \mathsf{K}$, this is well defined. Further, $\mathsf{SL}(d, \mathbb{R})$ acts by isometries:

$$\mathrm{d}_X(g \cdot g_1 \mathsf{K}, g \cdot g_2 \mathsf{K}) = \mathrm{d}_X(g_1 \mathsf{K}, g_2 \mathsf{K}).$$

Now

$$\log \sigma_j(g_1 \cdots g_n) \approx \lambda_j(\mu) n$$

and so the linear drift is given by

$$\lim_{n\to\infty} \frac{\mathrm{d}_X(\mathsf{K}, g_1 \cdots g_n \mathsf{K})}{n} = \sqrt{\sum_{j=1}^{d} \lambda_j(\mu)^2}.$$

One can show that $X$ is a manifold and the distance $\mathrm{d}_X$ is induced by a Riemannian metric $g$. Further the tangent space at $\mathsf{K}$ can be identified with

$$\mathfrak{p} := \{A \in M_d(\mathbb{R}) : \operatorname{tr} A = 0, \ A^t = A\}$$

in such a way that

- $\|A\|_g = \sqrt{\sum_{j=1}^{d} \sigma_j(A)^2}$.
- The Riemannian exponential map at $\mathsf{K}$ is a diffemorphism given by

$$\exp_{\mathsf{K}}(A) = e^A \,\mathsf{K},$$

  where $e^A$ is the matrix exponential.

This gives "polar coordinates" on $X$ where

$$\|A\|_g = \mathrm{d}_X(\mathsf{K}, e^A \,\mathsf{K})$$

is the radial component of $e^A \,\mathsf{K}$ and

$$\frac{A}{\|A\|_g}$$

is the angular component $e^A \,\mathsf{K}$.

Now if $g = ka\ell$, then

$$g \,\mathsf{K} = ka \,\mathsf{K} = kak^t \,\mathsf{K} = e^{k \log(a) k^t} \,\mathsf{K}$$

and $k \log(a) k^t \in \mathfrak{p}$. Using this, one can show that if $g_1 \cdots g_n = e^{A_n(\mathbf{g})} K$, then

$$\lim_{n\to\infty} \frac{A_n(\mathbf{g})}{\|A_n(\mathbf{g})\|_g}$$

converges almost surely. This means that "angular component" of the random walk converges almost surely.

10.2. **Ideas in the proof.** We now briefly discuss some of the main ideas.

**Idea #1:** We will fix a stationary measure on real projective space and then show that every limit measure is a Dirac mass. This will imply that

$$\lim_{n\to\infty} \frac{\sigma_p(g_1 \cdots g_n)}{\sigma_{p+1}(g_1 \cdots g_n)} = +\infty$$

almost surely and that

$$\lim_{n\to\infty} k_n(\mathbf{g}) \,\mathsf{M}$$

exists almost surely.

**Idea #2:** To prove that $\lambda_p(\mu) > \lambda_{p+1}(\mu)$ we need to show that $\log \frac{\sigma_p(g_1 \cdots g_n)}{\sigma_{p+1}(g_1 \cdots g_n)}$ converges to infinity with a linear lower bound. We accomplish this by using a trick from ergodic theory: if the Birkhoff sums diverge, then they diverge linearly (see Proposition 15.1 for a precise statement). This requires developing a dynamical system that we can apply the Birkhoff ergodic theorem to.

**Idea #3:** In the proof we will study $\lambda_1(\mu)$ and then use the wedge representations to obtain information about $\lambda_p(\mu)$ when $p > 1$. To make this work, we will need to assume a weaker condition than Zariski density, which we introduce in the next two sections.

## 11. Projective spaces and limits of transformations

In the proof we will study stationary measures on real projective space. Recall that real projective space is the quotient

$$\mathbb{P}(\mathbb{R}^d) = (\mathbb{R}^d - \{0\})/\sim$$

where $v \sim w$ if and only if $v = tw$ for some $t \in \mathbb{R}$. Geometrically, $\mathbb{P}(\mathbb{R}^d)$ parametrizes the set of one-dimensional linear subspaces in $\mathbb{R}^d$ (i.e. lines through the orgin).

Projective space with the quotient topology is a compact space and $[v_n] \to [v]$ if and only if there are lifts $\tilde{v}_n, \tilde{v} \in \mathbb{R}^d - \{0\}$ where $\tilde{v}_n \to \tilde{v}$. We use $[v_1 : \cdots : v_d]$ to denote the equivalence class of $(v_1, \ldots, v_d)$. We also let $\mathrm{d}_{\mathbb{P}}$ denote the angle metric on $\mathbb{P}(\mathbb{R}^d)$:

$$\mathrm{d}_{\mathbb{P}}([v],[w]) = \angle(\mathbb{R}\cdot v, \mathbb{R}\cdot w).$$

Notice that $\mathsf{GL}(d,\mathbb{R})$ acts continuously via the natural action

$$g \cdot [v] = [gv].$$

More generally, given $T \in \mathsf{End}(\mathbb{R}^d)$ we also let $T$ denote the continuous map

$$[v] \in \mathbb{P}(\mathbb{R}^d) \smallsetminus \mathbb{P}(\ker T) \mapsto T[v] = [Tv] \in \mathbb{P}(\mathbb{R}^d).$$

To study stationary measures on $\mathbb{P}(\mathbb{R}^d)$, it will be helpful to understand how sequences of elements in $\mathsf{GL}(d,\mathbb{R})$ act on $\mathbb{P}(\mathbb{R}^d)$.

**Example 11.1.** As a simple example, consider $g_n = \mathrm{diag}(2^n, 1, 2^{-n})$. Then

$$g_n[v_1 : v_2 : v_3] = [2^n v_1 : v_2 : 2^{-n} v_3].$$

So

$$\lim_{n\to\infty} g_n[v_1 : v_2 : v_3] = \begin{cases} [1:0:0] & \text{if } v_1 \neq 0 \\ [0:1:0] & \text{if } v_1 = 0, v_2 \neq 0 \\ [0:0:1] & \text{if } v_1 = v_2 = 0, v_3 \neq 0 \end{cases}.$$

For the general case, we exploit two observations

(1) The actions of $g$ and of $\frac{g}{\|g\|}$ on $\mathbb{P}(\mathbb{R}^d)$ are the same.
(2) The set $\{T : \|T\| \leq 1\}$ is compact in $\mathsf{End}(\mathbb{R}^d)$.

So given a sequence $\{g_n\} \subset \mathsf{GL}(d,\mathbb{R})$ we can pass to a subsequence so that $\frac{g_{n_j}}{\|g_{n_j}\|}$ converges to an element of $\mathsf{End}(\mathbb{R}^d)$.

**Observation 11.2.** If $T_n \to T$ in $\mathsf{End}(\mathbb{R}^d)$, then

$$T_n[v] \to T[v]$$

for all $[v] \in \mathbb{P}(\mathbb{R}^d) \smallsetminus \mathbb{P}(\ker T)$. Moreover, the convergence is uniform on any compact subset of $\mathbb{P}(\mathbb{R}^d) \smallsetminus \mathbb{P}(\ker T)$.

*Remark* 11.3. In Example 11.1, $\frac{g_n}{\|g_n\|} = \frac{g_n}{2^n}$ converges to $T = \mathrm{diag}(1,0,0)$, which has $\ker T = \langle e_2, e_3\rangle$ and $\mathrm{image}\, T = [e_1]$. So

$$g_n[v] \to T[v]$$

for all $[v] \in \mathbb{P}(\mathbb{R}^d) \smallsetminus \mathbb{P}(\ker T)$, is equivalent to $g_n[v_1 : v_2 : v_3] \to [1:0:0]$ whenever $v_1 \neq 0$.

*Proof.* Fix a compact set $K \subset \mathbb{P}(\mathbb{R}^d) \smallsetminus \mathbb{P}(\ker T)$. Suppose for a contradiction that the convergence is not uniform on $K$. Then we can find $\epsilon > 0$, $n_j \to \infty$, and $\{[v_{n_j}]\} \subset K$ such that

$$\mathrm{d}_{\mathbb{P}}(T_{n_j}[v_{n_j}], T[v_{n_j}]) > \epsilon.$$

By scaling we can assume that $\left\|v_{n_j}\right\| = 1$ for all $j$ and then passing to a subsequence we can suppose that $v_{n_j} \to v$. Then $[v] \in K$ and so $v \notin \ker T$. Then $T_{n_j} v_{n_j} \to Tv$ in $\mathbb{R}^d - \{0\}$ and so

$$T_{n_j}[v_{n_j}] = [T_{n_j} v_{n_j}] \to [Tv] = T[v].$$

However, we also have $Tv_{n_j} \to Tv$ and so $T[v_{n_j}] \to T[v]$. Then

$$\mathrm{d}_{\mathbb{P}}(T_{n_j}[v_{n_j}], T[v_{n_j}]) \to \mathrm{d}_{\mathbb{P}}(T[v], T[v]) = 0$$

and we have a contradiction. □

We extend the previous observation to convergence of measures.

**Observation 11.4.** Suppose $\nu$ is a probability measure on $\mathbb{P}(\mathbb{R}^d)$. If $T_n \to T$ in $\mathsf{End}(\mathbb{R}^d)$ and $\nu(\mathbb{P}(\ker T)) = 0$, then

$$(T_n)_* \nu \to T_* \nu.$$

*Remark* 11.5. The assumption $\nu(\mathbb{P}(\ker T)) = 0$ is necessary. For instance if $T_n = \mathrm{diag}(1, 2^{-n}, 2^{-n})$, then $T = \mathrm{diag}(1, 0, 0)$. So $(T_n)_* \delta_{[e_3]} = \delta_{[e_3]}$, but $T_* \delta_{[e_3]} \equiv 0$.

*Proof.* Fix a continuous function $\varphi : \mathbb{R}(\mathbb{R}^d) \to \mathbb{R}$. Then fix $\epsilon > 0$. Since $\nu(\mathbb{P}(\ker T)) = 0$, there exists an open neighborhood $U$ of $\mathbb{P}(\ker T)$ with $\nu(U) < \frac{\epsilon}{\|\varphi\|_\infty}$. For $n$ sufficiently large $\mathbb{P}(\ker T_n) \subset U$ and $\varphi \circ T_n \to \varphi \circ T$ uniformly on $K := \mathbb{P}(\mathbb{R}^d) \smallsetminus U$. So

$$\begin{aligned}\limsup_{n\to\infty} \int \varphi d(T_n)_* \nu &= \limsup_{n\to\infty} \int_{\mathbb{P}(\mathbb{R}^d) \smallsetminus \mathbb{P}(\ker T_n)} \varphi \circ T_n d\nu \leq \epsilon + \limsup_{n\to\infty} \int_K \varphi \circ T_n d\nu \\ &= \epsilon + \int_K \varphi \circ T d\nu \leq 2\epsilon + \int_{\mathbb{P}(\mathbb{R}^d) \smallsetminus \mathbb{P}(\ker T)} \varphi \circ T d\nu = 2\epsilon + \int \varphi dT_* \nu\end{aligned}$$

and likewise

$$\liminf_{n\to\infty} \int \varphi d(T_n)_* \nu \geq -2\epsilon + \int \varphi dT_* \nu.$$

Since $\epsilon > 0$ was arbitrary, $\int \varphi d(T_n)_* \nu \to \int \varphi dT_* \nu$.

□

11.1. **The singular value decomposition and limits in $\mathsf{End}(\mathbb{R}^d)$.** We have the following relationship between singular value decompositions and limits in $\mathsf{End}(\mathbb{R}^d)$.

**Proposition 11.6.** *Suppose $\{g_n\} \subset \mathsf{SL}(d, \mathbb{R})$ and $\frac{g_n}{\|g_n\|} \to T$ in $\mathsf{End}(\mathbb{R}^d)$. For each $n$ fix a singular value decomposition $g_n = k_n a_n \ell_n$. Then*

$$\begin{aligned}\operatorname{rank} T &= \max\left\{p : \lim_{n\to\infty} \frac{\sigma_p(g_n)}{\sigma_1(g_n)} > 0\right\}, \\ \operatorname{image} T &= \lim_{n\to\infty} k_n \left\langle e_1, \ldots, e_{\operatorname{rank} T}\right\rangle, \\ \ker T &= \lim_{n\to\infty} \ell_n^{-1} \left\langle e_{\operatorname{rank} T + 1}, \ldots, e_d\right\rangle.\end{aligned}$$

*Proof.* By continuity of the singular values, we have

$$\frac{\sigma_p(g_n)}{\sigma_1(g_n)} = \frac{\sigma_p(g_n)}{\|g_n\|} = \sigma_p\left(\frac{g_n}{\|g_n\|}\right) \to \sigma_p(T)$$

and so

$$\operatorname{rank} T = \max\{p : \sigma_p(T) > 0\} = \max\left\{p : \lim_{n\to\infty} \frac{\sigma_p(g_n)}{\sigma_1(g_n)} > 0\right\}.$$

To show that $k_n \langle e_1, \dots, e_{\operatorname{rank} T}\rangle$ converges to $\operatorname{image} T$, it suffices to show that any convergent subsequence converges to $\operatorname{image} T$. So fix $k_{n_j} \langle e_1, \dots, e_{\operatorname{rank} T}\rangle \to V \in \operatorname{Gr}_{\operatorname{rank} T}(\mathbb{R}^d)$. Passing to a subsequence we can assume $k_{n_j} \to k$ and $\ell_{n_j} \to \ell$. Then

$$T = \lim_{j\to\infty} \frac{g_{n_j}}{\|g_{n_j}\|} = \lim_{j\to\infty} \frac{k_{n_j} a_{n_j} \ell_{n_j}}{\|g_{n_j}\|} = k a' \ell$$

where $a' = \operatorname{diag}(1, \sigma_2(T), \dots, \sigma_d(T))$. Since $\sigma_k(T) > 0$ if and only if $k \le r$, we have

$$\operatorname{image} T = k \langle e_1, \dots, e_{\operatorname{rank} T}\rangle = V.$$

Thus $k_n \langle e_1, \dots, e_{\operatorname{rank} T}\rangle \to k \langle e_1, \dots, e_{\operatorname{rank} T}\rangle = \operatorname{image} T$.

The proof that $\ker T = \lim_{n\to\infty} \ell_n^{-1} \langle e_{\operatorname{rank} T+1}, \dots, e_d\rangle$ is very similar.

□

## 12. Proximal and strongly irreducible semigroups

In this section we introduce two important assumptions. In the proofs that follow, we will consider the semigroup generated by the support of our probability measure. Recall, that a subset $H$ of a group $G$ is a *semigroup* if

$$h_1, h_2 \in H \implies h_1 h_2 \in H.$$

12.1. **Proximal semigroups.** A semigroup $H \subset \mathsf{SL}(d, \mathbb{R})$ is *proximal* if there exists $\{h_n\} \subset H$ such that

$$\frac{h_n}{\|h_n\|} \to T$$

and $\operatorname{rank} T = 1$.

12.2. **Strongly irreducible (semi)groups.** We first define irreducible and strongly irreducible subgroups, then extend the definition to semigroups.

A subgroup $H \subset \mathsf{SL}(d, \mathbb{R})$ is *irreducible* if there are no proper $H$-invariant linear subspaces, i.e. the only $H$-invariant linear subspaces are $\{0\}$ and $\mathbb{R}^d$. A subgroup $H \subset \mathsf{SL}(d, \mathbb{R})$ is *strongly irreducible* if every finite index subgroup of $H$ is irreducible.

We want to generalize these notions to semigroups, but "finite index" does not extend well to semigroups. Instead we use the following characterization of strong irreducibility.

**Observation 12.1.** A subgroup $H \subset \mathsf{SL}(d, \mathbb{R})$ is strongly irreducible if and only if there are no finite union of proper linear subspaces which are invariant under $H$.

*Proof.* We prove that the negations are equivalent.

Suppose $H$ is not strongly irreducible. Then there exist a finite index subgroup $H_0 \subset H$ and a proper $H_0$-invariant linear subspace $V$. Let $\{h_1 H_0, \dots, h_m H_0\}$ be a set of coset representatives. Then $\cup h_j V$ is a finite union of proper linear subspaces which is invariant under $H$.

Suppose $\cup V_j$ is a finite union of proper linear subspaces which is invariant under $H$. Then $H$ permutes the $V_j$ and so there exists a finite index subgroup $H_0$ where each $V_j$ is $H_0$-invariant. □

**Definition 12.2.** A semigroup $H \subset \mathsf{SL}(d,\mathbb{R})$ is *irreducible* if there are no proper $H$-invariant linear subspaces and *strongly irreducible* if there are no finite union of proper linear subspaces which are invariant under $H$.

12.3. **Zariski closures.** We will use the following result from the theory of algebraic groups.

**Theorem 12.3** (Gol'dsheĭd–Margulis [GM89])**.** *Suppose $H \subset \mathsf{SL}(d,\mathbb{R})$ is a semigroup. If the group generated by $H$ is Zariski dense, then $H$ is strongly irreducible and proximal.*

*Proof.* We only prove that $H$ is strongly irreducible. For the proximaility, see [BQ16, Chapter 6].

Suppose for a contradiction that $\mathcal{V} = \cup_{j=1}^m V_j$ is a finite collection of proper linear subspaces which are invariant under $H$. We can assume that each $V_j$ has dimension $r$.

Then $h\,\mathcal{V} = \mathcal{V}$ for all $h \in H$ which implies that $\mathcal{V} = h^{-1}\,\mathcal{V}$ for all $h \in H$. Thus $g\,\mathcal{V} = \mathcal{V}$ for all $g \in \langle H\rangle$, the group generated by $H$.

For each $V_j$, fix a basis $b_{j,1},\dots,b_{j,r}$ and a set of linear functions $f_{j,1},\dots,f_{j,d-r}$ such that

$$\bigcap_{k=1}^{d-r} \ker f_{j,k} = V_j.$$

Then $gV_i = V_j$ if and only if $P_{i,j}(g) = \sum_{k,\ell} f_{j,k}(gb_{i,\ell})^2 = 0$. Let $P = \prod_j P_{1,j}$. Then $P|_{\langle H\rangle} \equiv 0$ which implies that $P|_{\mathsf{SL}(d,\mathbb{R})} \equiv 0$ (by Zariski density). Then for every $g \in \mathsf{SL}(d,\mathbb{R})$, we have

$$gV_1 \in \{V_1,\dots,V_m\}.$$

But this is clearly impossible since $\mathsf{SL}(d,\mathbb{R})$ acts transitively on the space of all $r$-dimensional linear spaces. □

## 13. Stationary measures on projective space

Let $G = \mathsf{SL}(d,\mathbb{R})$ and let $\mu$ be a probability measure on $\mathsf{SL}(d,\mathbb{R})$. As usual, let $\Omega = G^{\mathbb{N}}$ and $\beta = \mu^{\otimes\mathbb{N}}$.

Let $\Gamma_\mu$ be the semigroup generated by $\operatorname{supp}(\mu)$, i.e.

$$\Gamma_\mu = \bigcup_{m=1}^{\infty} \operatorname{supp}(\mu)^m.$$

**Theorem 13.1.** *If $\Gamma_\mu$ is strongly irreducible and proximal, then:*

1. *There is a unique stationary measure $\nu$ on $\mathbb{P}(\mathbb{R}^d)$.*
2. *If $V \subset \mathbb{R}^d$ is a proper linear subspace, then $\nu(\mathbb{P}(V)) = 0$.*
3. *There exist a Borel map $\zeta : \Omega \to \mathbb{P}(\mathbb{R}^d)$ such that $\nu = \zeta_*\beta$ and for almost every $\mathbf{g} \in \Omega$*
   (a) *the limit measure $\nu_{\mathbf{g}}$ is a Dirac mass centered at $\zeta(\mathbf{g})$.*

*(b) every limit point of*

$$\frac{g_1 \cdots g_n}{\|g_1 \cdots g_n\|}$$

*in* $\mathsf{End}(\mathbb{R}^d)$ *has rank one and image* $\zeta(\mathbf{g})$.

As a corollary to (3) we have the following.

**Corollary 13.2.** *For almost every* $\mathbf{g} \in \Omega$,

$$\lim_{n\to\infty} \frac{\sigma_1(g_1 \cdots g_n)}{\sigma_2(g_1 \cdots g_n)} = +\infty$$

*and if* $g_1 \cdots g_n = k_n a_n \ell_n$ *is a singular value decomposition then*

$$\lim_{n\to\infty} k_n[e_1] = \zeta(\mathbf{g}).$$

*Proof.* Recall from Proposition 11.6 that if $\frac{h_n}{\|h_n\|} \to T$ in $\mathsf{End}(\mathbb{R}^d)$, then

$$\operatorname{rank} T = \max\left\{p : \lim_{n\to\infty} \frac{\sigma_p(h_n)}{\sigma_1(h_n)} > 0\right\},$$
$$\operatorname{image} T = \lim_{n\to\infty} k_n \langle e_1, \ldots, e_{\operatorname{rank} T}\rangle$$

where $h_n = k_n a_n \ell_n$ is a singular value decomposition. So the corollary follows immediately from Theorem 13.1 (3). □

**Lemma 13.3.** *If* $\nu$ *is a stationary measure and* $V \subset \mathbb{R}^d$ *is a proper linear subspace, then* $\nu(\mathbb{P}(V)) = 0$.

*Proof.* Suppose not. Let $r$ be the minimal dimension where there exists a $r$ dimensional linear space linear subspace $V \subset \mathbb{R}^d$ such that $\nu(\mathbb{P}(V)) > 0$. Then

$$\nu(\mathbb{P}(V_1) \cap \mathbb{P}(V_2)) = 0$$

whenever $V_1, V_2$ are distinct and both have dimension $r$. So if $\{V_j\}$ is a collection of distinct $r$ dimensional linear subspaces, then

$$\sum_j \nu(\mathbb{P}(V_j)) = \nu\left(\bigcup_j \mathbb{P}(V_j)\right) \leq 1. \tag{1}$$

Let $c := \sup_{\dim V = r} \nu(\mathbb{P}(V))$. By Equation (1) there are only finitely many subspaces with measure at least $c/2$. Hence $\mathcal{V} := \{V : \dim V = r, \nu(\mathbb{P}(V)) = c\}$ is non-empty and finite.

**Claim:** If $V \in \mathcal{V}$ and $g \in \operatorname{supp}(\mu)$, then $g^{-1}V \in \mathcal{V}$.

Suppose $\nu(\mathbb{P}(V)) = c$. Then

$$c = \nu(\mathbb{P}(V)) = (\mu * \nu)(\mathbb{P}(V)) = \int_G \nu(\mathbb{P}(g^{-1}V))d\mu(g) \leq \int_G c d\mu(g) = c.$$

Thus $g^{-1}V \in \mathcal{V}$ for $g$ in a full $\mu$-measure set $A \subset G$. If $g \in \operatorname{supp}(\mu)$, then there exists $(g_n) \subset A$ with $g_n \to g$. Then

$$g^{-1}V = \lim_{n\to\infty} g_n^{-1}V.$$

Since $\{g_n^{-1}V\} \subset \mathcal{V}$ and $\mathcal{V}$ is finite, the sequence eventually stabilizes. Hence $g^{-1}V \in \mathcal{V}$. Thus the claim is established.

Now $g^{-1}\mathcal{V} \subset \mathcal{V}$ for all $g \in \Gamma_\mu$, which implies that $g^{-1}\mathcal{V} = \mathcal{V}$ since $\mathcal{V}$ is finite. Thus $\mathcal{V} = g\mathcal{V}$ for all $g \in \Gamma_\mu$. But by assumption $\Gamma_\mu$ is strongly irreducible and so we have a contradiction. □

**Lemma 13.4.** *If $\nu$ is a stationary measure, then there exists a Borel map $\zeta : \Omega \to \mathbb{P}(\mathbb{R}^d)$ such that for $\beta$-almost every $\mathbf{g} \in \Omega$*

*(1) the limit measure $\nu_{\mathbf{g}}$ is a Dirac mass centered at $\zeta(\mathbf{g})$.*
*(2) every limit point of*

$$\frac{g_1 \cdots g_n}{\|g_1 \cdots g_n\|}$$

*in $\mathsf{End}(\mathbb{R}^d)$ has image $\zeta(\mathbf{g})$.*

*Proof.* By Lemma 13.3 and Observation 11.4: if $T_n \to T$ in $\mathsf{End}(\mathbb{R}^d)$ and $T \neq 0$, then

$$(T_n)_*\nu \to T_*\nu. \tag{2}$$

By Corollary 9.6, for every $m \in \mathbb{Z}_{\geq 0}$, for $\mu^{*m} \otimes \beta$-almost every $(h, \mathbf{g})$ one has

$$(g_1 \cdots g_n h)_*\nu \to \nu_{\mathbf{g}}.$$

(here we assume $\mu^{*0}$ is the Dirac mass at the identity). Hence, there exists a full measure set $\Omega' \subset \Omega$ such that for every $\mathbf{g} \in \Omega'$, for every $m \in \mathbb{Z}_{\geq 0}$, and for $\mu^{*m}$-almost every $h \in G$ we have

$$(g_1 \cdots g_n h)_*\nu \to \nu_{\mathbf{g}}.$$

Fix $\mathbf{g} \in \Omega'$. Let $T$ be a limit point of $\left\{ \frac{g_1 \cdots g_n}{\|g_1 \cdots g_n\|} \right\}$ in $\mathsf{End}(\mathbb{R}^d)$. Then

$$(Th)_*\nu = \nu_{\mathbf{g}}$$

for every $m \in \mathbb{Z}_{\geq 0}$ and for $\mu^{*m}$-almost every $h \in G$. Hence, by Equation (2),

$$(Th)_*\nu = \nu_{\mathbf{g}}$$

for every $h \in \Gamma_\mu$.

Since $\Gamma_\mu$ is proximal, there exists $\{h_n\} \subset \Gamma_\mu$ where $\frac{h_n}{\|h_n\|} \to S$ and $\operatorname{rank} S = 1$. Since $\Gamma_\mu$ is irreducible, we can pick $h \in \Gamma_\mu$ such that $h \operatorname{image} S \not\subset \ker T$. Then $Thh_n \to ThS \neq 0$ and so

$$(ThS)_*\nu = \lim_{n\to\infty} (Thh_n)_*\nu = \lim_{n\to\infty} \nu_{\mathbf{g}} = \nu_{\mathbf{g}}.$$

Since $ThS$ has rank one, $\nu_{\mathbf{g}}$ is a Dirac mass centered at $\zeta(\mathbf{g}) := \operatorname{image}(ThS)$. This proves part (1).

To prove part (2), it suffices to show that $T$ has rank one. Notice that

$$1 = \nu_{\mathbf{g}}\left(\zeta(\mathbf{g})\right) = T_*\nu\left(\zeta(\mathbf{g})\right) = \nu\big(T^{-1}\left(\zeta(\mathbf{g})\right)\big).$$

Now $T^{-1}(\zeta(\mathbf{g}))$ is contained in the projectivization of a subspace of dimension $1 + \dim(\ker T)$ and so Lemma 13.3 implies that $\dim \ker T = d - 1$, i.e. $T$ has rank one.

□

**Lemma 13.5.** *There is a unique stationary measure $\nu$ and if $\zeta$ is the map in Lemma 13.4, then $\nu = \zeta_*\beta$.*

*Proof.* Suppose $\nu_1, \nu_2$ are two stationary measures. Let $\zeta_1, \zeta_2 : \Omega \to \mathbb{P}(\mathbb{R}^d)$ be the maps in Lemma 13.4. Then for $j = 1, 2$ and for $\beta$-almost every $\mathbf{g} \in \Omega$ every limit point of

$$\frac{g_1 \cdots g_n}{\|g_1 \cdots g_n\|}$$

in $\mathsf{End}(\mathbb{R}^d)$ has image $\zeta_j(\mathbf{g})$. Hence $\zeta_1 = \zeta_2$ $\beta$-almost everywhere. Further,

$$\nu_1 = \int (\nu_1)_{\mathbf{g}} d\beta(\mathbf{g}) = \int \delta_{\zeta_1(\mathbf{g})} d\beta(\mathbf{g}) = \int \delta_{\zeta_2(\mathbf{g})} d\beta(\mathbf{g}) = \int (\nu_2)_{\mathbf{g}} d\beta(\mathbf{g}) = \nu_2.$$

Thus there is a unique stationary unique stationary measure $\nu$. Further, if $\zeta$ is the map in Lemma 13.4, then

$$\nu(A) = \int \delta_{\zeta(\mathbf{g})}(A) d\beta(\mathbf{g}) = \int 1_A(\zeta(\mathbf{g})) d\beta(\mathbf{g}) = \int 1_{\zeta^{-1}A}(\mathbf{g}) d\beta(\mathbf{g}) = \beta(\zeta^{-1}A) = \zeta_*\beta(A).$$

So $\nu = \zeta_*\beta$. □

## 14. First steps towards the proof of Theorem 10.1

Let $G = \mathsf{SL}(d, \mathbb{R})$ and let $\mu$ be a probability measure on $\mathsf{SL}(d, \mathbb{R})$. As usual, let $\Omega = G^{\mathbb{N}}$ and $\beta = \mu^{\otimes \mathbb{N}}$.

Using Theorem 13.1 we can prove part of Theorem 10.1.

**Theorem 14.1.** *If*

- $\int \log \|g\| \, d\mu(g) < +\infty$,
- *the group generated by* $\operatorname{supp}(\mu)$ *is Zariski dense in* $\mathsf{SL}(d, \mathbb{R})$,

*then*

$$\lim_{n\to\infty} \frac{\sigma_p(g_1 \cdots g_n)}{\sigma_{p+1}(g_1 \cdots g_n)} = +\infty$$

*almost surely. Moreover, if* $g_1 \cdots g_n = k_n a_n \ell_n$ *is a singular value decomposition, then the limit*

$$\lim_{n\to\infty} k_n \, \mathsf{M}$$

*exist almost surely in* $\mathsf{SO}(d)/\mathsf{M}$.

Consider the wedge representation $\tau_p : \mathsf{SL}(d, \mathbb{R}) \to \mathsf{SL}(\wedge^p \mathbb{R}^d)$. If we identify $\wedge^p \mathbb{R}^d$ with $\mathbb{R}^{\dim \wedge^p \mathbb{R}^d}$ via the standard basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p}\}$, then the singular values of $\tau_p(g)$ are

$$\{\sigma_{i_1}(g) \cdots \sigma_{i_p}(g) : i_1 < \cdots < i_p\}.$$

In particular,

$$\sigma_1(\tau_p(g)) = \sigma_1(g) \cdots \sigma_p(g) \quad \text{and} \quad \sigma_2(\tau_p(g)) = \sigma_1(g) \cdots \sigma_{p-1}(g)\sigma_{p+1}(g).$$

So

$$\frac{\sigma_1(\tau_p(g))}{\sigma_2(\tau_p(g))} = \frac{\sigma_p(g)}{\sigma_{p+1}(g)}.$$

Further, if $g = ka\ell$ is a singular value decomposition, then $\tau_p(k), \tau_p(\ell) \in \mathsf{SO}(\dim \wedge^p \mathbb{R}^d)$ and $\tau_p(a)$ is diagonal with entries $\{\sigma_{i_1}(g) \cdots \sigma_{i_p}(g) : i_1 < \cdots < i_p\}$. So $\tau_p(g)$ has singular value decomposition

$$\tau_p(k) Q^{-1} [Q \tau_p(a) Q^{-1}] Q \tau_p(\ell)$$

where $Q = Q(p, g)$ is a signed permutation matrix which arranges the entries of $\tau_p(a)$ is decreasing order (notice that we need the word "signed" so that $\det Q = 1$).

Next we use the following result.

**Theorem 14.2** (Gol'dsheĭd–Margulis [GM89]). *Suppose $S \subset \mathsf{SL}(d,\mathbb{R})$ and the group generated by $S$ is Zariski dense in $\mathsf{SL}(d,\mathbb{R})$. If $\Gamma_S$ is the semigroup generated by $S$, then for $k = 1,\dots,d-1$ the semigroup $\tau_k(\Gamma_S) \subset \mathsf{SL}(\wedge^k \mathbb{R}^d)$ is proximal and strongly irreducible.*

*Proof.* One can prove strong irreducibility as we did in the proof of Theorem 12.3. For the proximality, see [BQ16, Chapter 6]. □

So we can apply Corollary 13.2 to the measure $(\tau_p)_*\mu$. Then almost surely

$$\lim_{n\to\infty} \frac{\sigma_p(g_1\cdots g_n)}{\sigma_{p+1}(g_1\cdots g_n)} = \lim_{n\to\infty} \frac{\sigma_1(\tau_p(g_1)\cdots\tau_p(g_n))}{\sigma_2(\tau_p(g_1)\cdots\tau_p(g_n))} = +\infty$$

and if $g_1\cdots g_n = k_n a_n \ell_n$ is a singular value decomposition

$$\lim_{n\to\infty} k_n[e_1\wedge\cdots\wedge e_p] = \zeta^p(\mathbf{g}),$$

where $\zeta^p : \Omega \to \mathbb{P}(\wedge^p \mathbb{R}^d)$ is a Borel map.

Now fix $\mathbf{g}$ where $k_n[e_1\wedge\cdots\wedge e_p] \to \zeta^p(\mathbf{g})$ for $p = 1,\dots,d-1$. We claim that $\lim_{n\to\infty} k_n \mathsf{M}$ exists. Let $u_p(\mathbf{g})$ be a unit vector orthogonal to $\zeta^{p-1}(\mathbf{g})$ in $\zeta^p(\mathbf{g})$. Then any limit point of $k_n e_p$ is $\pm u_k(\mathbf{g})$. Hence if $k(\mathbf{g}) \in \mathsf{SO}(d)$ is an element with $k(\mathbf{g})e_p = \pm u_p(\mathbf{g})$, then

$$\lim_{n\to\infty} k_n \mathsf{M} = k(\mathbf{g})\,\mathsf{M}\,.$$

## 15. Linear Divergence of Birkhoff Sums

To finish the proof of Theorem 10.1, we need to upgrade the divergence

$$\lim_{n\to\infty} \frac{\sigma_p(g_1\cdots g_n)}{\sigma_{p+1}(g_1\cdots g_n)} = +\infty$$

in Theorem 14.1 to show that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\sigma_p(g_1\cdots g_n)}{\sigma_{p+1}(g_1\cdots g_n)} > 0.$$

This will be accomplished by using a trick from ergodic theory: if the Birkhoff sums for an ergodic system diverge to infinity, then they diverge linearly. We follow the proof in [BQ16, Section 3.5].

**Proposition 15.1.** *Suppose $T : (X,\lambda) \to (X,\lambda)$ is ergodic. Let $f \in L^1(X,\lambda)$ and let $f_n := f + f\circ T + \cdots + f\circ T^{n-1}$. Then the following are equivalent:*

*(1)* $\int f d\lambda > 0$.
*(2)* $\lim_{n\to\infty} \frac{1}{n} f_n(x) > 0$ *for almost everywhere.*
*(3)* $\lim_{n\to\infty} f_n(x) = +\infty$ *almost everywhere,*

*Proof.* By the Birkhoff ergodic theorem, $\lim_{n\to\infty} \frac{1}{n} f_n(x) = \frac{1}{\lambda(X)} \int f d\lambda$ almost everywhere, so (1) ⇔ (2). Also, clearly (2) ⇒ (3). So it suffices to prove that (3) ⇒ (1).

Suppose for a contradiction that $\lim_{n\to\infty} f_n(x) = +\infty$ for almost every $x$, but $\int f d\lambda \le 0$. If $\int f d\lambda < 0$, then the Birkhoff ergodic theorem would imply that $\lim_{n\to\infty} f_n(x) = -\infty$ for almost every $x$ and so we must have $\int f d\lambda = 0$.

By assumption there exists $N > 0$ such that the set

$$A := \{x : f_n(x) \ge 1 \text{ for all } n \ge N\}$$

has positive measure. Applying the Birkhoff ergodic theorem to $f$ and $1_A$, we can fix $x \in X$ such that

$$\frac{1}{n} f_n(x) \to \frac{1}{\lambda(X)} \int f d\lambda = 0$$

and

$$\frac{1}{n+1} \sum_{m=0}^{n} 1_A(T^m x) \to \frac{1}{\lambda(X)} \int 1_A d\lambda = \frac{\lambda(A)}{\lambda(X)}.$$

Let $E := \{m : T^m x \in A\}$. Then

$$\frac{|E \cap [0,n]|}{n+1} = \frac{1}{n+1} \sum_{m=0}^{n} 1_A(T^m x) \to \frac{\lambda(A)}{\lambda(X)}.$$

Next fix a maximal subset

$$\{0 \le m_1 < m_2 < \cdots\}$$

of $E$ where $m_{j+1} \ge m_j + N$ for all $j = 1, 2, \ldots$. By maximality,

$$E \subset \bigcup_{j=1}^{\infty} [m_j - N, m_j + N].$$

Hence

$$|E \cap [0, m_j]| \le (2N+1)j.$$

Next notice that

$$f_{m_j}(x) = f_{m_{j-1}}(x) + f_{m_j - m_{j-1}}(T^{m_{j-1}} x) \ge f_{m_{j-1}}(x) + 1$$

and so by induction

$$f_{m_j}(x) \ge (j-1) + f_{m_1}(x)$$

(where $f_0 \equiv 0$ if $m_1 = 0$). Then

$$0 = \lim_{j\to\infty} \frac{f_{m_j}(x)}{m_j} \ge \limsup_{j\to\infty} \frac{j}{m_j} \ge \frac{1}{2N+1} \limsup_{j\to\infty} \frac{|E \cap [0, m_j]|}{m_j} = \frac{\lambda(A)}{(2N+1)\lambda(X)} > 0$$

and we have a contradiction.

□

## 16. An integral formula

In this section we setup the ergodic system and function that we will apply Proposition 15.1 to.

Let $G = \mathsf{SL}(d, \mathbb{R})$ and let $\mu$ be a probability measure on $\mathsf{SL}(d, \mathbb{R})$ where

$$\int_{\mathsf{SL}(d,\mathbb{R})} \log \|g\| \, d\mu(g) < +\infty.$$

Also suppose that the semigroup $\Gamma_\mu$ generated by $\mathrm{supp}(\mu)$ is strongly irreducible and proximal. Then by Theorem 13.1 there exists a unique stationary measure $\nu$ on $\mathbb{P}(\mathbb{R}^d)$.

As usual, let $\Omega = G^{\mathbb{N}}$ and $\beta = \mu^{\otimes \mathbb{N}}$. Then let $S : \Omega \times \mathbb{P}(\mathbb{R}^d) \to \Omega \times \mathbb{P}(\mathbb{R}^d)$ be the extended shift map

$$S(\mathbf{g}, [v]) = (T\mathbf{g}, g_1[v]).$$

Since $\nu$ is the unique stationary measure, it is ergodic. So Proposition 8.5 implies that $S$ is ergodic with respect to $\beta \otimes \nu$.

Define $f : \Omega \times \mathbb{P}(\mathbb{R}^d) \to \mathbb{R}$ by

$$f(\mathbf{g}, [v]) = \log \frac{\|g_1 v\|}{\|v\|}.$$

Then

$$\int f d(\beta \otimes \nu) = \int_{\mathsf{SL}(d,\mathbb{R}) \times \mathbb{P}(\mathbb{R}^d)} \log \frac{\|gv\|}{\|v\|} d(\mu \otimes \nu)(g, [v]).$$

Further, the Birkhoff sums satisfy

$$\sum_{j=0}^{n-1} f \circ S^j(\mathbf{g}, [v]) = \sum_{j=0}^{n-1} \log \frac{\|g_{j+1} \cdots g_1 v\|}{\|g_j \cdots g_1 v\|} = \log \frac{\|g_n \cdots g_1 v\|}{\|v\|}.$$

**Theorem 16.1.**

$$\lambda_1(\mu) = \int f d(\beta \otimes \nu) = \int_{\mathsf{SL}(d,\mathbb{R}) \times \mathbb{P}(\mathbb{R}^d)} \log \frac{\|gv\|}{\|v\|} d(\mu \otimes \nu)(g, [v]).$$

*Hence, by ergodicity,*

$$\lim_{n \to \infty} \frac{1}{n} \log \|g_n \cdots g_1 v\| = \lambda_1(\mu).$$

*for $\beta \otimes \nu$-almost every $(\mathbf{g}, [v])$.*

*Proof.* Let

$$I := \int f d(\beta \otimes \nu).$$

We need to show that $I = \lambda_1(\mu)$. By the definition of Lyapunov exponents, for $\beta$-almost every $\mathbf{g}$ we have

$$\lambda_1(\mu) = \lim_{n \to \infty} \frac{1}{n} \log \sigma_1(g_n \cdots g_1) = \lim_{n \to \infty} \frac{1}{n} \log \|g_n \cdots g_1\|.$$

By the ergodicity of $S$, for $\beta$-almost every $\mathbf{g}$ we have

$$\frac{1}{n} \log \frac{\|g_n \cdots g_1 v\|}{\|v\|} \to I$$

for $\nu$-almost every $[v]$. Fix $\mathbf{g}$ satisfying both almost everywhere conditions.

Fix $n_j \to \infty$ where

$$\frac{g_{n_j} \cdots g_1}{\sigma_1(g_{n_j} \cdots g_1)} = \frac{g_{n_j} \cdots g_1}{\|g_{n_j} \cdots g_1\|} \to T$$

in $\mathsf{End}(\mathbb{R}^d)$. By Theorem 13.1, $\nu(\mathbb{P}(\ker T)) = 0$ and hence we can fix $[v] \in \mathbb{P}(\mathbb{R}^d)$ such that $[v] \notin \mathbb{P}(\ker T)$ and

$$\frac{1}{n} \log \frac{\|g_n \cdots g_1 v\|}{\|v\|} \to I.$$

Then

$$\begin{aligned} I &= \lim_{j \to \infty} \frac{1}{n_j} \log \frac{\|g_{n_j} \cdots g_1 v\|}{\|v\|} = \lim_{j \to \infty} \frac{1}{n_j} \log \|g_{n_j} \cdots g_1\| + \frac{1}{n_j} \log \left\| \frac{g_{n_j} \cdots g_1 v}{\|g_{n_j} \cdots g_1\|} \right\| \\ &= \lambda_1(\mu) + 0 \cdot \|T(v)\| = \lambda_1(\mu). \end{aligned}$$

□

## 17. Reverse products

The Birkhoff sums for the function introduced in the last section involve the random products $g_n \cdots g_1$ (instead of the usual random products $g_1 \cdots g_n$). The next result studies such products.

Suppose $\mu$ is a probability measure on $G = \mathsf{SL}(d, \mathbb{R})$ where the semigroup $\Gamma_\mu$ generated by $\mathrm{supp}(\mu)$ is strongly irreducible and proximal. Let $\Omega = G^{\mathbb{N}}$ and $\beta = \mu^{\otimes \mathbb{N}}$ be as usual.

**Theorem 17.1.**

*(1)* $\lim_{n\to\infty} \frac{\sigma_1(g_n \cdots g_1)}{\sigma_2(g_n \cdots g_1)} = +\infty$ *almost surely.*

*(2) There exist a Borel map* $\zeta^* : \Omega \to \mathrm{Gr}_{d-1}(\mathbb{R}^d)$ *such that for almost every* $\mathbf{g} \in \Omega$ *every limit point of*

$$\frac{g_n \cdots g_1}{\|g_n \cdots g_1\|}$$

*in* $\mathsf{End}(\mathbb{R}^d)$ *has kernel* $\zeta^*(\mathbf{g})$.

*(3) If* $[v] \in \mathbb{P}(\mathbb{R}^d)$, *then*

$$\beta(\{\mathbf{g} : v \in \zeta^*(\mathbf{g})\}) = 0.$$

*(4) If* $[v] \in \mathbb{P}(\mathbb{R}^d)$, *then for almost every* $\mathbf{g} \in \Omega$ *we have*

$$\liminf_{n\to\infty} \frac{\|g_n \cdots g_1 v\|}{\|g_n \cdots g_1\| \, \|v\|} > 0.$$

*Proof.* Define $\tau : \mathsf{SL}(d, \mathbb{R}) \to \mathsf{SL}(d, \mathbb{R})$ by $\tau(g) = g^t$. Let $\mu^t := \tau_* \mu$ and $\beta^t := (\mu^t)^{\otimes \mathbb{N}}$. Then $\Gamma_{\mu^t} = \tau(\Gamma_\mu)$.

If $\{g_n\} \subset \Gamma_\mu$ and $\frac{g_n}{\|g_n\|} \to T$ where $T$ has rank one, then $\frac{g_n^t}{\|g_n^t\|} = \left(\frac{g_n}{\|g_n\|}\right)^t \to T^t$ and $T^t$ has rank one. Hence $\Gamma_{\mu^t}$ is proximal. If $\mathcal{V} := \cup V_i$ is a finite union of subspaces and $\Gamma_{\mu^t} \cdot \mathcal{V} = \mathcal{V}$, then $\Gamma_\mu \cdot \mathcal{V}' = \mathcal{V}'$ where $\mathcal{V}' := \cup V_i^\perp$. So $\Gamma_{\mu^t}$ is strongly irreducible.

Then Theorem 13.1 has the following consequences:

- For $\beta^t$-almost every $\mathbf{h} \in G^{\mathbb{N}}$,
$$\lim_{n\to\infty} \frac{\sigma_1(h_1 \cdots h_n)}{\sigma_2(h_1 \cdots h_n)} = +\infty.$$
- There exists a Borel map $\zeta_0 : \Omega \to \mathbb{P}(\mathbb{R}^d)$ such that for $\beta^t$-almost every $\mathbf{h} \in \Omega$ every limit point of
$$\frac{h_1 \cdots h_n}{\|h_1 \cdots h_n\|}$$
in $\mathsf{End}(\mathbb{R}^d)$ has rank one and image $\zeta_0(\mathbf{h})$.
- $\nu^t = (\zeta_0)_* \beta^t$ is the unique stationary measure.
- $\nu^t$ assigns zero mass to every proper projective subspace.

Then for $\beta$-almost every $\mathbf{g} \in \Omega$,

$$\lim_{n\to\infty} \frac{\sigma_1(g_1^t \cdots g_n^t)}{\sigma_2(g_1^t \cdots g_n^t)} = +\infty$$

and every limit point of

$$\frac{g_1^t \cdots g_n^t}{\|g_1^t \cdots g_n^t\|}$$

in $\mathsf{End}(\mathbb{R}^d)$ has rank one and image $\zeta_0(\tau\mathbf{g})$, where we abuse notation and also let $\tau$ denote the map

$$(g_1, g_2, \dots) \in \Omega \mapsto (g_1^t, g_2^t, \dots) \in \Omega.$$

Now

$$\sigma_k(g_n \cdots g_1) = \sigma_k((g_n \cdots g_1)^t) = \sigma_k(g_1^t \cdots g_n^t)$$

and so (1) follows.

Further, if $T$ is a limit point of $\frac{g_n \cdots g_1}{\|g_n \cdots g_1\|}$, then $T^t$ is a limit point of

$$\frac{g_1^t \cdots g_n^t}{\|g_1^t \cdots g_n^t\|} = \frac{(g_n \cdots g_1)^t}{\|g_n \cdots g_1\|}$$

and $\ker T = (\operatorname{image} T^t)^\perp$. So (2) follows with $\zeta^*(\mathbf{g}) = \zeta_0(\tau\mathbf{g})^\perp$.

To prove (3) and (4), fix $[v] \in \mathbb{P}(\mathbb{R}^d)$. Then $v \in \zeta^*(\mathbf{g})$ if and only if $\zeta_0(\tau\mathbf{g}) \subset v^\perp$. Then, since $\nu^t$ assigns zero mass to every proper projective subspace,

$$\begin{aligned}\beta(\{\mathbf{g} : v \in \zeta^*(\mathbf{g})\}) &= \beta(\{\mathbf{g} : \zeta_0(\tau\mathbf{g}) \subset v^\perp\}) \\ &= (\zeta_0\tau)_*\beta(v^\perp) = (\zeta_0)_*\beta^t(v^\perp) = \nu^t(v^\perp) = 0.\end{aligned}$$

This proves (3). Now for $\beta$-almost every $\mathbf{g}$, we have $v \notin \zeta^*(\mathbf{g})$ and every limit point of

$$\frac{g_n \cdots g_1}{\|g_n \cdots g_1\|}$$

in $\mathsf{End}(\mathbb{R}^d)$ has kernel $\zeta^*(\mathbf{g})$. For such $\mathbf{g}$, fix $n_j \to \infty$ such that

$$L := \liminf_{n\to\infty} \frac{\|g_n \cdots g_1 v\|}{\|g_n \cdots g_1\| \, \|v\|} = \lim_{j\to\infty} \frac{\|g_{n_j} \cdots g_1 v\|}{\|g_{n_j} \cdots g_1\| \, \|v\|}.$$

Passing to a subsequence we can suppose $\frac{g_{n_j} \cdots g_1}{\|g_{n_j} \cdots g_1\|} \to T$ in $\mathsf{End}(\mathbb{R}^d)$. Then by assumption $v \notin \zeta^*(\mathbf{g}) = \ker T$. Then $L = \frac{\|T(v)\|}{\|v\|} > 0$. □

## 18. Proof of Theorem 10.1

Let $G = \mathsf{SL}(d, \mathbb{R})$ and let $\mu$ be a probability measure on $\mathsf{SL}(d, \mathbb{R})$ with

$$\int_{\mathsf{SL}(d,\mathbb{R})} \log \|g\| \, d\mu(g) < +\infty.$$

As usual, let $\Omega = G^{\mathbb{N}}$ and $\beta = \mu^{\otimes \mathbb{N}}$.

Theorem 10.1 is a consequence of Theorem 14.1 and the following.

**Theorem 18.1.** *If the group generated by* $\operatorname{supp}(\mu)$ *is Zariski dense in* $\mathsf{SL}(d, \mathbb{R})$*, then*

$$\lambda_1(\mu) > \lambda_2(\mu) > \cdots > \lambda_d(\mu).$$

Consider the wedge representation $\tau_p : \mathsf{SL}(d, \mathbb{R}) \to \mathsf{SL}(\wedge^p \mathbb{R}^d)$. Then by the proof of Theorem 14.1

(1) $\sigma_1(\tau_p(g)) = \sigma_1(g) \cdots \sigma_p(g)$,
(2) $\sigma_2(\tau_p(g)) = \sigma_1(g) \cdots \sigma_{p-1}(g)\sigma_{p+1}(g)$,
(3) $\tau_p(\Gamma_\mu)$ is proximal and strongly irreducible,
(4) $\lambda_1(\mu_p) = \lambda_1(\mu) + \cdots + \lambda_p(\mu)$ where $\mu_p := (\tau_p)_*\mu$.

Applying Theorem 17.1 to $\mu_p$ we have

$$\lim_{n\to\infty} \frac{\sigma_p(g_n \cdots g_1)}{\sigma_{p+1}(g_n \cdots g_1)} = \lim_{n\to\infty} \frac{\sigma_1(\tau_p(g_n \cdots g_1))}{\sigma_2(\tau_p(g_n \cdots g_1))} = +\infty \tag{3}$$

almost surely and for every $[v] \in \mathbb{P}(\wedge^p \mathbb{R}^d)$

$$0 < \liminf_{n\to\infty} \frac{\|\tau_p(g_n \cdots g_1)v\|}{\|\tau_p(g_n \cdots g_1)\| \, \|v\|} = \liminf_{n\to\infty} \frac{\|\tau_p(g_n \cdots g_1)v\|}{\sigma_1(g_n \cdots g_1) \cdots \sigma_p(g_n \cdots g_1) \, \|v\|} \tag{4}$$

almost surely.

By Theorem 13.1 there is a unique stationary measure $\nu_p$ on $\mathbb{P}(\wedge^p \mathbb{R}^d)$. Further, by Theorem 16.1, if $f_p : \Omega \times \mathbb{P}(\wedge^p \mathbb{R}^d) \to \mathbb{R}$ is the function

$$f_p(\mathbf{g}, [v]) = \log \frac{\|\tau_p(g)v\|}{\|v\|}$$

then

$$\int f_p d(\beta \otimes \nu_p) = \lambda_1(\mu_p) = \lambda_1(\mu) + \cdots + \lambda_p(\mu).$$

Fix $p \in \{1, \ldots, d-1\}$. Let

$$Y := \mathbb{P}(\wedge^{p-1} \mathbb{R}^d) \times \mathbb{P}(\wedge^p \mathbb{R}^d) \times \mathbb{P}(\wedge^{p+1} \mathbb{R}^d)$$

(when $p = 1$ we take $Y = \mathbb{P}(\mathbb{R}^d) \times \mathbb{P}(\wedge^2 \mathbb{R}^d)$ and modify the proof accordingly). Then $\mathsf{SL}(d, \mathbb{R})$ acts on $Y$ by

$$g \cdot ([v_1], [v_2], [v_3]) = (\tau_{p-1}(g)[v_1], \tau_p(g)[v_2], \tau_{p+1}(g)[v_3]).$$

Define $f : \Omega \times Y \to \mathbb{R}$ by

$$f(\mathbf{g}, ([v_1], [v_2], [v_3])) = 2f_p(\mathbf{g}, [v_2]) - f_{p-1}(\mathbf{g}, [v_1]) - f_{p+1}(\mathbf{g}, [v_3]).$$

Finally, fix an ergodic stationary measure $\nu$ on $Y$ (which exists by Remark 8.2).

**Lemma 18.2.** $\int f d(\beta \otimes \nu) = \lambda_p(\mu) - \lambda_{p+1}(\mu)$.

*Proof.* For $i = -1, 0, 1$, let $\pi_i : Y \to \mathbb{P}(\wedge^{p+i} \mathbb{R}^d)$ be the natural projection. Then $\tau_{p+i}(g) \circ \pi_i = \pi_i \circ g$ for all $g \in \mathsf{SL}(d, \mathbb{R})$. Then

$$\mu_{p+i} * (\pi_i)_* \nu = \int \tau_{p+i}(g)_* (\pi_i)_* \nu d\mu(g) = (\pi_i)_* \int g_* \nu d\mu(g) = (\pi_i)_* \nu.$$

Hence $(\pi_i)_* \nu$ is stationary and thus, by uniqueness, $(\pi_i)_* \nu = \nu_{p+i}$. Thus

$$\begin{aligned} \int f d(\beta \otimes \nu) &= 2 \int f_p d(\beta \otimes \nu_p) - \int f_{p-1} d(\beta \otimes \nu_{p-1}) - \int f_{p+1} d(\beta \otimes \nu_{p+1}) \\ &= \lambda_p(\mu) - \lambda_{p+1}(\mu). \end{aligned}$$

□

Let $S : \Omega \times Y \to \Omega \times Y$ be the extended shift map $S(\mathbf{g}, y) = (T\mathbf{g}, g_1 \cdot y)$. Since $\nu$ is ergodic, Proposition 8.5 implies that $S$ is ergodic with respect to $\beta \otimes \nu$.

**Lemma 18.3.** *For every $y \in Y$ and $\beta$-almost every $\mathbf{g}$,*

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} f \circ S^j(\mathbf{g}, y) = +\infty.$$

*Hence, by Proposition 15.1,*

$$0 < \int f d(\beta \otimes \nu) = \lambda_p(\mu) - \lambda_{p+1}(\mu).$$

*Proof.* Notice that

$$\sum_{j=0}^{n-1} f \circ S^j(\mathbf{g}, y) = \log \frac{\|\tau_p(g_n \cdots g_1)v_2\|^2}{\|\tau_{p-1}(g_n \cdots g_1)v_1\| \, \|\tau_{p+1}(g_n \cdots g_1)v_3\|} \frac{\|v_1\| \, \|v_3\|}{\|v_2\|^2}.$$

So by Equation (4) for every $y \in Y$ and almost every $\mathbf{g}$ there exists $C = C(y, \mathbf{g}) > 0$ such that

$$\sum_{j=0}^{n-1} f \circ S^j(\mathbf{g}, y) \geq -C + \log \frac{\sigma_p(g_n \cdots g_1)}{\sigma_{p+1}(g_n \cdots g_1)}.$$

Then by Equation (3) for every $y \in Y$ and $\beta$-almost every $\mathbf{g}$,

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} f \circ S^j(\mathbf{g}, y) = +\infty. \qquad \square$$

## Part 3. Random Walks on Gromov Hyperbolic Spaces

### 19. Statement of Results

In this part we consider random walks on Gromov hyperbolic metric spaces (defined in Section 20 below).

Suppose $X$ is a Gromov hyperbolic metric space and $\mu$ is a probability measure on the isometry group $\mathsf{Isom}(X)$. The metric space $X$ admits a natural compactification $X \sqcup \partial_{\mathcal{G}} X$ (defined in Section 20 below) and $\mathsf{Isom}(X)$ extends to a continuous action on this compactification.

A semigroup $H \subset \mathsf{Isom}(X)$ is *non-elementary* if

(1) There are no $H$-invariant subsets in $\partial_{\mathcal{G}} X$ with cardinality 1 or 2.
(2) Every (some) orbit of $H$ in $X$ is unbounded.

We can think of (1) as a type of irreducibility assumption and (2) as a type of proximality assumption.

In this part, we will prove the following results.

**Theorem 19.1.** *Suppose the semigroup generated by* $\mathrm{supp}(\mu)$ *is non-elementary. Then:*

*(1) If $o \in X$, then*

$$\lim_{n \to \infty} g_1 \cdots g_n(o)$$

*exists and is in $\partial_{\mathcal{G}} X$ for $\beta$-almost every $\mathbf{g}$.*

*(2) There is a unique stationary measure $\nu$ on $\partial_{\mathcal{G}} X$ and for any measurable $A \subset \partial_{\mathcal{G}} X$ we have*

$$\nu(A) = \beta(\{\mathbf{g} : \lim_{n \to \infty} g_1 \cdots g_n o \in A\}).$$

*(3) If $\int_{\mathsf{Isom}(X)} \mathrm{d}(o, go) d\mu(g) < +\infty$ for some (any) $o \in X$, then the linear drift is positive.*

For more properties of random walks in this context, see Maher–Tiozzo [MT18] (who prove much more and consider non-proper Gromov hyperbolic spaces).

We will prove the above theorem by adapting our previous arguments for matrix groups. In particular, we will use stationary measures to obtain qualitative information and then upgrade this using ergodic theory.

## 20. Gromov hyperbolic metric spaces

In this section we recall the definition of a Gromov hyperbolic metric space and some of their basic properties.

Suppose $X$ is a metric space.

- A *geodesic in* $X$ is a map $\ell : I \to X$ where $I \subset \mathbb{R}$ is an interval and
$$\mathrm{d}(\ell(s), \ell(t)) = |s - t| \quad \forall s, t \in I.$$
- $X$ is *proper* if every bounded set is relatively compact and *geodesic* if every two points are joined by a geodesic.
- A *geodesic triangle* in $X$ is a choice of three points and geodesics joining them.
- A geodesic triangle $T$ is *$\delta$-slim* if every point on $T$ is contained in the $\delta$-neighborhood of the other two sides.

**Definition 20.1.** A proper geodesic metric space $(X, \mathrm{d})$ is *$\delta$-hyperbolic* if every geodesic triangle is $\delta$-slim. A metric space is *Gromov hyperbolic* if it is $\delta$-hyperbolic for some $\delta \geq 0$.

**Example 20.2.** If $d \geq 2$, then $\mathbb{R}^d$ with the Euclidean metric is not Gromov hyperbolic.

*Proof.* For $R > 0$, consider the triangle $T_R$ with vertices $0$, $(R, 0, \ldots, 0)$, and $(0, R, 0, \ldots, 0)$. There exists some $\epsilon > 0$ such that $T_1$ is not $\epsilon$-slim. Then $T_R$ is not $R\epsilon$-slim. So $\mathbb{R}^d$ is not Gromov hyperbolic. □

**Example 20.3.** If $T$ is a (possibly infinite) tree with the geodesic distance making every edge isometric to $[0, 1]$, then $T$ is 0-hyperbolic.

*Proof.* Notice that in any geodesic triangle any side is contained in the union of the other two sides. So $T$ is 0-hyperbolic. □

**Example 20.4.** The hyperbolic plane with the Poincaré metric is Gromov hyperbolic. Recall, that the hyperbolic plane is $\mathbb{H}^2 = \{z : \mathrm{Im}(z) > 0\}$ endowed with the Riemannian metric $\frac{dx^2+dy^2}{y^2}$. Then the distance is given by
$$\rho(z_1, z_2) = \inf_\gamma \int_a^b \frac{|\gamma'(t)|}{\mathrm{Im}(\gamma(t))} dt$$
where the infimum is taken over all piecewise smooth curves $\gamma : [a, b] \to \mathbb{H}^2$ joining $z_1$ to $z_2$.

*Proof.* The proof requires two facts:

- Geodesics in $\mathbb{H}^2$ are either vertical lines or half circles which intersect the real line at right angles,
- $\mathsf{Isom}(\mathbb{H}^2)$ acts transitively on $\mathbb{H}^2$.

For $z_1, z_2 \in \mathbb{H}^2$ let $[z_1, z_2]$ denote the geodesic joining $z_1$ to $z_2$.

Suppose for a contradiction that for every $n \geq 1$ there exists a geodesic triangle with vertices $x_n, y_n, z_n$ and $u_n \in [x_n, y_n]$ where
$$\rho(u_n, [y_n, z_n] \cup [z_n, x_n]) > n.$$

Translating by $\mathsf{Isom}(\mathbb{H}^2)$ we can assume that $u_n = i$ for all $n$. Then passing to a subsequence we can suppose that $x_n, y_n, z_n \to x, y, z$. Since

$$\lim_{n\to\infty} \rho(u_n, [y_n, z_n] \cup [z_n, x_n]) = +\infty$$

we must have $x, y, z \in \mathbb{R} \cup \{\infty\}$. The geodesic line joining $x$ and $y$ contains $i$ and so we must have $x \neq y$. Thus either $z \neq x$ or $z \neq y$. By possibly relabelling we can assume that $x \neq z$. Then fix $v$ in the geodesic line joining $x$ to $z$. Then we can find $v_n \in [x_n, z_n]$ with $v_n \to v$. Then

$$+\infty = \lim_{n\to\infty} \rho(u_n, [y_n, z_n] \cup [z_n, x_n]) \leq \lim_{n\to\infty} \rho(u_n, v_n) = \rho(i, v) < +\infty.$$

So we have a contradiction.

□

**Theorem 20.5.** *Suppose $\Gamma$ is a finitely generated group. Then either every Cayley graph of $\Gamma$ is Gromov hyperbolic or every Cayley graph of $\Gamma$ is not Gromov hyperbolic.*

**Definition 20.6.** A finitely generated group is *word hyperbolic* if its Cayley graphs are Gromov hyperbolic.

**Example 20.7.** A non-abelian free group is word hyperbolic.

**Example 20.8.** The fundamental group of a closed Riemannian manifold with negative sectional curvature is word hyperbolic (e.g. a surface with genus at least two).

20.1. **The Gromov boundary.** In this section we describe a compactification of a Gromov hyperbolic metric space $X$.

**Definition 20.9.**

(1) Two geodesic rays $\ell_1, \ell_2 : [0, \infty) \to X$ are *equivalent* if there exists $R > 0$ such that

$$\mathrm{d}(\ell_1(t), \ell_2(t)) \leq R \quad \text{for all} \quad t \geq 0.$$

(2) The *Gromov boundary* denoted by $\partial_{\mathcal{G}} X$ is the set of equivalence classes of geodesic rays $\ell : [0, \infty) \to X$.

One can show that $\partial_{\mathcal{G}} X$ naturally compactifies $X$.

**Theorem 20.10.** *The set $\overline{X}^{\mathcal{G}} := X \sqcup \partial_{\mathcal{G}} X$ has a natural topology which makes it a compactification of $X$, that is: $\overline{X}^{\mathcal{G}}$ is a compact metrizable space, the inclusion map $X \hookrightarrow \overline{X}^{\mathcal{G}}$ is a homeomorphism onto its image, and $X$ is open dense in $\overline{X}^{\mathcal{G}}$.*

For a proof see [BH99, Part III, Chapter H, Section 3].

*Remark* 20.11. The proof of the theorem is somewhat technical, but convergence in the topology is easy to describe. First notice that by the Arzelá–Ascoli theorem, if $\{\ell_n\}$ is a sequence of geodesics with $\{\ell_n(0)\}$ bounded, then $\{\ell_n\}$ is relatively compact in the topology of local uniform convergence and any limit point is a geodesic.

- A sequence $\{x_n\} \subset X$ converges to $\xi \in \partial_G X$ if and only if whenever $\ell_n : [0, b_n] \to X$ are geodesics with $\{\ell_n(0)\}$ bounded and $\ell_n(b_n) = x_n$, then $b_n \to \infty$ and any limit point of $\{\ell_n\}$ is a geodesic ray in the equivalence class of $\xi$. In particular, if $\ell : [0, \infty) \to X$ is a geodesic ray, then
$$\lim_{t\to\infty} \ell(t) = [\ell].$$
- A sequence $\{\xi_n\} \subset \partial_G X$ converges to $\xi \in \partial_\infty X$ if and only if whenever $\ell_n : [0, +\infty) \to X$ are geodesic rays with $\{\ell_n(0)\}$ bounded and $[\ell_n] = \xi_n$, then any limit point of $\{\ell_n\}$ is a geodesic ray in the equivalence class of $\xi$.

Isometries map geodesics to geodesics and hence we also have the following.

**Proposition 20.12.** *If $f : X \to X$ is an isometry, then $f$ extends continuously to a homeomorphism $\overline{X}^{\mathcal{G}} \to \overline{X}^{\mathcal{G}}$. Moreover, with this extension, $\mathsf{Isom}(X)$ acts continuously on $\overline{X}^{\mathcal{G}}$.*

**Example 20.13.**

- The Gromov boundary of the hyperbolic plane is $\partial_{\mathcal{G}} \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$.
- The Gromov boundary of an infinite tree is the space of ends.

20.2. **Visibility and Contraction.** In this subsection we recall an important property of Gromov hyperbolic spaces (sometimes called the visibility lemma) and then use it to prove a contracting property for sequences of isometries.

Suppose for the rest of the subsection that $(X, \mathrm{d})$ is $\delta$-hyperbolic.

**Proposition 20.14** (Visibility Lemma)**.** *Suppose $\xi^+, \xi^- \in \partial_{\mathcal{G}} X$ are distinct. Then*

*(1) For any neighborhoods $V^\pm$ of $\xi^\pm$ with $\overline{V^+} \cap \overline{V^-} = \emptyset$, there exists a compact set $K \subset X$ such that any geodesic joining a point in $V^-$ to a point in $V^+$ intersects $K$.*

*(2) There exists a geodesic line $\ell : \mathbb{R} \to X$ with $\lim_{t\to\pm\infty} \ell(t) = \xi^\pm$.*

*Proof.* (1). Suppose (1) is false. Fix $o \in X$. Then for every $n$ there exist $x_n^\pm \in V^\pm \cap X$ and a geodesic segment $\ell_n$ joining $x_n^\pm$ so that $\mathrm{d}(\ell_n, o) > n$. Let $\ell_n^\pm$ be a geodesic joining $o$ to $x_n^\pm$. Passing to a subsequence we can suppose that $\ell_n^\pm$ converges to a geodesic ray $\ell^\pm : [0, \infty) \to X$. Then $[\ell^\pm] \in \overline{V^\pm}$ and hence $\ell^+, \ell^-$ are in different equivalence classes. So
$$\limsup_{t\to\infty} \mathrm{d}(\ell^-(t), \ell^+(t)) = +\infty.$$
So we can fix $t_0 > 0$ where $\mathrm{d}(\ell^-(t_0), \ell^+(t_0)) > 2\delta$. Passing to a tail of our sequences we can assume that
$$\mathrm{d}(\ell_n^-(t_0), \ell_n^+(t_0)) > 2\delta$$
for all $n \geq 1$.

Now consider the geodesic triangles $\ell_n^- \cup \ell_n^+ \cup \ell_n$. Since $X$ is Gromov hyperbolic,
$$\mathrm{d}(\ell_n^+(t_0), \ell_n^- \cup \ell_n) \leq \delta.$$
Notice that
$$\mathrm{d}(\ell_n^+(t_0), \ell_n^-(s)) \geq \left|\mathrm{d}(\ell_n^+(t_0), o) - \mathrm{d}(o, \ell_n^-(s))\right| = |t_0 - s|$$
and
$$\mathrm{d}(\ell_n^+(t_0), \ell_n^-(s)) \geq \left|\mathrm{d}(\ell_n^+(t_0), \ell_n^-(t_0)) - \mathrm{d}(\ell_n^-(t_0), \ell_n^-(s))\right| > 2\delta - |t_0 - s|.$$

So

$$\mathrm{d}(\ell_n^+(t_0), \ell_n^-) > \delta.$$

Thus

$$\mathrm{d}(\ell_n^+(t_0), \ell_n) \leq \delta.$$

Thus

$$n \leq \mathrm{d}(o, \ell_n) \leq \delta + t_0$$

and so we have a contradiction.

(2). Fix $\{x_n^\pm\} \subset X$ such that $x_n^\pm \to \xi^\pm$. Let $\ell_n$ be a geodesic joining $x_n^\pm$. By (1), each $\ell_n$ intersects a fixed compact set $K$. So we can pass to a subsequence and reparametrize so that $\ell_n$ converges locally uniformly to a geodesic line $\ell : \mathbb{R} \to X$. By Remark 20.11, $\ell|_{[0,\infty)}$ is in the equivalence class of $\xi^+$ and so $\lim_{t\to\infty} \ell(t) = \xi^+$. Likewise, $\lim_{t\to-\infty} \ell(t) = \xi^-$. □

**Corollary 20.15.** *If $\{x_n\} \subset X$ converges to $\xi \in \partial_{\mathcal{G}} X$ and $\{y_n\} \subset X$ satisfies*

$$R := \sup_{n\geq 1} \mathrm{d}(x_n, y_n) < +\infty,$$

*then $y_n \to \xi$.*

*Proof.* Fix $o \in X$ and a geodesic $\ell_n$ joining $x_n$ to $y_n$. Then

$$\mathrm{d}(o, \ell_n) \geq \mathrm{d}(o, x_n) - R \to +\infty$$

So for any compact set $K \subset X$ we eventually have $\ell_n \cap K = \emptyset$. Thus, by the visibility lemma, $y_n \to \xi$.

□

The Gromov boundary has the following important contraction property with respect to isometries.

**Theorem 20.16.** *Suppose $\{\gamma_n\} \subset \mathrm{Isom}(X)$ is a sequence of isometries and $o \in X$. If $\gamma_n o \to \xi^+ \in \partial_{\mathcal{G}} X$ and $\gamma_n^{-1} o \to \xi^- \in \partial_{\mathcal{G}} X$, then*

$$\gamma_n x \to \xi^+$$

*for all $x \in \overline{X}^{\mathcal{G}} \smallsetminus \{\xi^-\}$ and the convergence is uniform on compact subsets of $\overline{X}^{\mathcal{G}} \smallsetminus \{\xi^-\}$.*

*Proof.* Fix a compact set $K \subset \overline{X}^{\mathcal{G}} \smallsetminus \{\xi^-\}$ and suppose for a contradiction that $\gamma_n|_K$ does not converge uniformly to the constant map $\xi^+$. Then after passing to a subsequence there exist a neighborhood $U$ of $\xi^+$ and $\{x_n\} \subset K$ such that $\gamma_n x_n \notin U$ for all $n \geq 1$. By perturbing, we can assume that $\{x_n\} \subset X$.

Fix a geodesic $\ell_n : [0, b_n] \to X$ joining $o$ to $\gamma_n x_n$. Passing to a subsequence we can suppose that $\ell_n \to \ell$ locally uniformly. Since $\gamma_n^{-1}\ell_n$ joins $\gamma_n^{-1} o$ to $x_n$, by the visibility property, there exists $\{t_n\}$ such that

$$R := \sup_{n\geq 1} \mathrm{d}(\gamma_n^{-1}\ell_n(t_n), o) < +\infty.$$

Then

$$\mathrm{d}(\ell_n(t_n), \gamma_n o) \leq R.$$

Since $\gamma_n o \to \xi^+ \in \partial_{\mathcal{G}} X$, we must have $t_n \to +\infty$. Thus $b_n \to +\infty$ and $\ell$ is a geodesic ray.

Since $\ell_n$ joins $o$ to $\gamma_n x_n$, we have

$$[\ell] = \lim_{n\to\infty} \gamma_n x_n \notin U$$

and hence $[\ell] \neq \xi^+$. On the other hand, $\ell_n|_{[0,t_n]} \to \ell$ and so

$$[\ell] = \lim_{n\to\infty} \ell(t_n).$$

Then by the corollary, $[\ell] = \xi^+$. So we have a contradiction. □

**Corollary 20.17.** *Suppose $\{\gamma_n\} \subset \mathrm{Isom}(X)$ is a sequence of isometries and $o \in X$. If $\gamma_n o \to \xi^+ \in \partial_{\mathcal{G}} X$, $\gamma_n^{-1} o \to \xi^- \in \partial_{\mathcal{G}} X$, and $\nu$ is a probability measure on $\partial_{\mathcal{G}} X$ with $\nu(\{\xi^-\}) = 0$, then*

$$(\gamma_n)_*\nu \to \delta_{\xi^+}.$$

*Proof.* This is very similar to the proof of Observation 11.4. □

## 21. Stationary Measures

Suppose $X$ is a Gromov hyperbolic space and $\mu$ is a probability measure on $\mathsf{Isom}(X)$.

Recall that a semigroup $H \subset \mathsf{Isom}(X)$ is *non-elementary* if

(1) There are no $H$-invariant subsets in $\partial_{\mathcal{G}} X$ with cardinality 1 or 2.
(2) Every (some) orbit of $H$ in $X$ is unbounded.

**Theorem 21.1.** *If the semigroup $\Gamma_\mu^+$ generated by $\mathrm{supp}(\mu)$ is non-elementary, then:*

*(1) There is a unique stationary measure $\nu$ on $\partial_{\mathcal{G}} X$.*
*(2) $\nu$ has no atoms.*
*(3) There exists a Borel map $\zeta : \Omega \to \partial_{\mathcal{G}} X$ such that for $\beta$-almost every $\mathbf{g}$:*
  *(a) The limit measure $\nu_{\mathbf{g}}$ is a Dirac mass centered at $\zeta(\mathbf{g})$.*
  *(b) $\lim_{n\to\infty} g_1 \cdots g_n(o) = \zeta(\mathbf{g})$ for all $o \in X$.*
*(4) If $o \in X$ and $A \subset \partial_{\mathcal{G}} X$ is measurable, then*

$$\nu(A) = \Pr\Big(\lim_{n\to\infty} g_1 \cdots g_n(o) \in A\Big) = \beta(\{\mathbf{g} : \zeta(\mathbf{g}) \in A\}),$$

*equivalently $\nu = \zeta_*\beta$.*

*Remark* 21.2. Recall from Theorem 9.1, that given a stationary measure $\nu$ the limit

$$\nu_{\mathbf{g}} = \lim_{n\to\infty} (g_1 \cdots g_n)_*\nu$$

exists for $\beta$-almost every $\mathbf{g}$ and

$$\nu = \int \nu_{\mathbf{g}} d\beta.$$

The proof is very similar to the proof of Theorem 13.1.

**Lemma 21.3.** *If $\nu$ is a stationary measure, then $\nu$ has no atoms.*

*Proof.* Suppose not. Let $c := \sup\{\nu(\{y\}) : y \in \partial_{\mathcal{G}} X\}$. Since $\nu$ is a probability measure the set of points with mass at least $c/2$ is finite and so the set $F := \{y : \nu(\{y\}) = c\}$ is finite and non-empty. Since $\nu$ is stationary,

$$\nu(\{y\}) = \int_{\mathsf{Isom}(X)} \nu(\{g^{-1}y\}) d\mu(g).$$

So if $y \in F$, then $g^{-1}y \in F$ for $\mu$-almost every $g$. If $y \in F$ and $g \in \mathrm{supp}(\mu)$, then there exists $g_n \to g$ where $g_n^{-1}y \in F$ for all $n$. Since $F$ is finite, then $g^{-1}y \in F$.

Thus $g^{-1}F = F$ for all $g \in \mathrm{supp}(\mu)$. Hence $F$ is $\Gamma_\mu^+$-invariant. Then by assumption $|F| \geq 3$.

Fix $o \in X$. By assumption there exists $\{g_n\} \subset \Gamma_\mu^+$ such that $\mathrm{d}(g_n(o), o) \to +\infty$. Passing to a subsequence we can assume that $g_n o \to \xi^+$ and $g_n^{-1} o \to \xi^-$. Then $g_n x \to \xi^+$ for all $x \in \overline{X}^{\mathcal{G}} \smallsetminus \{\xi^-\}$. Since $|F| \geq 3$, we can fix distinct $a, b \in F - \{\xi^-\}$. Then $g_n a \to \xi^+$ and $g_n b \to \xi^+$. Further, for every $n \geq 1$ at least one of $g_n a$ and $g_n b$ does not equal $\xi^+$. So we can find $\{x_n\} \subset F$ such that $x_n \to \xi^+$ and $x_n \neq \xi^+$ for all $n$. Thus $\{x_n\}$ contains infinitely many distinct points, which contradicts the fact that $F$ is finite. □

**Lemma 21.4.** *There exist a Borel map $\zeta : \Omega \to \partial_{\mathcal{G}} X$ such that:*

*(1) If $o \in X$, then $\lim_{n\to\infty} g_1 \cdots g_n(o) = \zeta(\mathbf{g})$ for $\beta$-almost every $\mathbf{g}$.*
*(2) $\nu = \zeta_* \beta$ is the unique stationary measure and $\nu_{\mathbf{g}}$ is a Dirac mass centered at $\zeta(\mathbf{g})$ for $\beta$-almost every $\mathbf{g}$.*

*Proof.* Fix a stationary measure $\nu$. By Corollary 9.6, for every $m \in \mathbb{Z}_{\geq 0}$, for $\mu^{*m} \otimes \beta$-almost every $(h, \mathbf{g})$ one has

$$(g_1 \cdots g_n h)_* \nu \to \nu_{\mathbf{g}}$$

(here we define $\mu^{*0} = \delta_{\mathrm{id}}$). Hence, there exists a full measure set $\Omega' \subset \Omega$ such that for every $\mathbf{g} \in \Omega'$, for every $m \in \mathbb{Z}_{\geq 0}$, and for $\mu^{*m}$-almost every $h \in \mathsf{Isom}(X)$ we have

$$(g_1 \cdots g_n h)_* \nu \to \nu_{\mathbf{g}}.$$

Fix $\mathbf{g} \in \Omega'$ and fix $o \in X$. We first claim that every limit point of $\{(g_1 \cdots g_n)(o)\}$ is in $\partial_{\mathcal{G}} X$. Suppose not. Then there exists $n_j \to \infty$ such that $(g_1 \dots g_{n_j})(o) \to x \in X$. Using the Arzelá-Ascoli theorem and passing to a subsequence we can suppose that $(g_1 \dots g_{n_j}) \to T$ in $\mathsf{Isom}(X)$. Then

$$\nu_{\mathbf{g}} = \lim_{j\to\infty} (g_1 \cdots g_{n_j} h)_* \nu = T_* h_* \nu \tag{5}$$

for every $m \in \mathbb{Z}_{\geq 0}$ and for $\mu^{*m}$-almost every $h \in \mathsf{Isom}(X)$. Then by continuity, $T_* h_* \nu = \nu_{\mathbf{g}}$ for every $h \in \Gamma_\mu^+$. Since $\Gamma_\mu^+$ is non-elementary, there exists $\{h_n\} \subset \Gamma_\mu^+$ with $\mathrm{d}(h_n o, o) \to +\infty$. Passing to a subsequence we can assume that $h_n o \to \xi^+ \in \partial_{\mathcal{G}} X$ and $h_n^{-1} o \to \xi^- \in \partial_{\mathcal{G}} X$. Then

$$h_n x \to \xi^+$$

for all $x \in \overline{X}^G \smallsetminus \{\xi^-\}$. Since $\nu$ is atomless, Corollary 20.17 implies

$$\nu_{\mathbf{g}} = \lim_{n\to\infty} T_* (h_n)_* \nu = T_* \delta_{\xi^+} = \delta_{T\xi^+}.$$

By Equation (5), we have also $T_* \nu = \nu_{\mathbf{g}}$ and so

$$\nu = (T^{-1})_* \nu_{\mathbf{g}} = (T^{-1})_* \delta_{T\xi^+} = \delta_{\xi^+}$$

is a Dirac mass, which contradicts the fact that $\nu$ is atomless. Thus every limit point of $\{(g_1 \cdots g_n)(o)\}$ is in $\partial_{\mathcal{G}} X$.

Now suppose $\eta^+$ is a limit point of $\{(g_1 \cdots g_n)(o)\}$. Then there exists $n_j \to \infty$ such that $(g_1 \dots g_{n_j})(o) \to \eta^+$. Passing to a further subsequence, we can suppose that $(g_1 \dots g_{n_j})^{-1}(o) \to \eta^-$. Then, since $\nu$ is atomless, Corollary 20.17 implies

$$\nu_{\mathbf{g}} = \lim_{j\to\infty} (g_1 \dots g_{n_j})_* \nu = \delta_{\eta^+}.$$

This implies that $\eta^+$ is the only limit point of $\{(g_1 \cdots g_n)(o)\}$. Hence if $\zeta(\mathbf{g}) := \eta^+$, then

$$\lim_{n\to\infty} (g_1 \cdots g_n)(o) = \zeta(\mathbf{g}).$$

Since $\mathbf{g} \in \Omega'$ was arbitrary,

$$\zeta(\mathbf{g}) := \lim_{n\to\infty} (g_1 \cdots g_n)(o)$$

exists for all $\mathbf{g} \in \Omega'$ and $\Omega'$ has full $\beta$-measure.

Now suppose $\tilde{\nu}$ is a (possibly different) stationary measure. There exists a full measure set $\Omega''$ such that

$$\tilde{\nu}_{\mathbf{g}} = \lim_{n\to\infty} (g_1 \cdots g_n)_* \tilde{\nu}$$

exists for all $\mathbf{g} \in \Omega''$. If $\mathbf{g} \in \Omega' \cap \Omega''$, then $(g_1 \cdots g_n)(o) \to \zeta(\mathbf{g})$ and so Corollary 20.17 implies that

$$\tilde{\nu}_{\mathbf{g}} = \lim_{n\to\infty} (g_1 \cdots g_n)_* \tilde{\nu} = \delta_{\zeta(\mathbf{g})}.$$

Then, since $\Omega' \cap \Omega''$ has full measure,

$$\tilde{\nu} = \int \tilde{\nu}_{\mathbf{g}} d\beta(\mathbf{g}) = \int \delta_{\zeta(\mathbf{g})} d\beta(\mathbf{g}),$$

i.e. $\tilde{\nu} = \zeta_* \beta$. Hence $\zeta_* \beta$ is the unique stationary measure.

□

## 22. The Busemann boundary and an integral formula for the linear drift

In this section we establish a result of Karlsson–Ledrappier, see Theorem 18 in [KL11], which provides an integral formula for the linear drift.

Suppose $X$ is a proper metric space. Fix a point $o \in X$ and let $\mathcal{K} \subset C(X)$ be the set of 1-Lipschitz functions with $f(o) = 0$. Define a distance $\rho$ on $\mathcal{K}$ by

$$\rho_0(f, g) = \sum_{n=1}^{\infty} 2^{-n} \max_{x:\mathrm{d}(x,o)\le n} |f(x) - g(x)|.$$

Then $(\mathcal{K}, \rho_0)$ is a compact metric space where $f_n \to f$ with respect to $\rho_0$ if and only if $f_n$ converges uniformly on compact sets to $f$.

We map $X$ into $\mathcal{K}$ as follows. Given $y \in X$, let $b_y : X \to \mathbb{R}$ be the function

$$b_y(x) = \mathrm{d}(x, y) - \mathrm{d}(o, y).$$

Then $b_y \in \mathcal{K}$. The *Busemann boundary*, denoted $\partial_B X$, is the set of all functions $b : X \to \mathbb{R}$ where $b = \lim_{n\to\infty} b_{y_n}$ for some sequence $\{y_n\} \subset X$ with $\mathrm{d}(o, y_n) \to \infty$.

**Observation 22.1.** The set $\overline{X}^B := X \sqcup \partial_B X$ admits a topology which makes it a compactification of $X$, i.e. $\overline{X}^B$ is a compact metrizable space, the inclusion map $X \hookrightarrow \overline{X}^B$ is a homeomorphism onto its image, and $X$ is open dense in $\overline{X}^B$. Further

(1) $\{x_n\} \subset X$ converges to $b \in \partial_B X$ if and only if $\mathrm{d}(o, x_n) \to \infty$ and $b_{x_n} \to b$ locally uniformly.
(2) The action of $\mathsf{Isom}(X)$ on $X$ extends continuously to an action on $\overline{X}^B$ where $\mathsf{Isom}(X)$ acts on $\partial_B X$ by

$$g \cdot b = b \circ g^{-1} - b(g^{-1}o).$$

*Proof.* Let $h(x) = \frac{1}{1+\mathrm{d}(o,x)}$. Then define a distance $\rho$ on $\overline{X}^B$ by

$$\begin{aligned}\rho(x,y) &= \min\{\mathrm{d}(x,y), h(x)+h(y)\} + \rho_0(b_x, b_y) \quad \text{when} \quad x,y \in X,\\ \rho(x,b) &= \rho(b,x) = h(x) + \rho_0(b_x, b) \quad \text{when} \quad x \in X,\ b \in \partial_B X,\\ \rho(b_1,b_2) &= \rho_0(b_1,b_2) \quad \text{when} \quad b_1, b_2 \in \partial_B X.\end{aligned}$$

Check that this has all the desired properties.

□

Next we define $c_B : \mathsf{Isom}(X) \times \overline{X}^B \to \mathbb{R}$ by

$$c_B(g,b) = \begin{cases} b(g^{-1}o) & \text{if } b \in \partial_B X \\ \mathrm{d}(g^{-1}o, b) - \mathrm{d}(o,b) & \text{if } b \in X \end{cases}$$

Notice that $c_B$ is continuous.

**Observation 22.2.** $|c_B(g,\cdot)| \le \mathrm{d}(go,o)$ and $c_B$ is a cocycle, i.e.

$$c_B(g_1g_2, b) = c_B(g_1, g_2b) + c_B(g_2, b)$$

for all $g_1, g_2 \in \mathsf{Isom}(X)$ and $b \in \overline{X}^B$.

*Proof.* By density and continuity, we can assume that $b \in X$. Then

$$|c_B(g,b)| = \left|\mathrm{d}(g^{-1}o, b) - \mathrm{d}(o,b)\right| \le \mathrm{d}(g^{-1}o, o) = \mathrm{d}(o, go)$$

and

$$\begin{aligned} c_B(g_1g_2, b) &= \mathrm{d}(g_2^{-1}g_1^{-1}o, b) - \mathrm{d}(o,b)\\ &= \mathrm{d}(g_1^{-1}o, g_2b) - \mathrm{d}(o, g_2b) + \mathrm{d}(g_2^{-1}o, b) - \mathrm{d}(o,b)\\ &= c_B(g_1, g_2b) + c_B(g_2, b).\end{aligned}$$

□

**Theorem 22.3.** *If $X$ is a proper metric space, $\mu$ is a probability measure on $G := \mathsf{Isom}(X)$, and*

$$\int_G \mathrm{d}(o, go) d\mu(g) < +\infty$$

*(hence the linear drift exists), then there exists an ergodic stationary measure $\nu$ on $\overline{X}^B$ such that the linear drift satisfies*

$$L_\mu = \int_{G \times \overline{X}^B} c_B d(\mu \otimes \nu).$$

*Moreover, if $\mathrm{d}(o, g_1 \cdots g_n(o)) \to +\infty$ almost surely, then $\mathrm{supp}(\nu) \subset \partial_B X$.*

For the rest of the section suppose $\mu$ satisfies the hypothesis of the theorem.

**Lemma 22.4.** *If $\nu$ is a stationary measure on $\overline{X}^B$, then*

$$\int_{G \times \overline{X}^B} c_B d(\mu \otimes \nu) \le L_\mu.$$

*Proof.* Since $\nu$ is stationary, for any $k \geq 1$

$$\begin{aligned}\int_{G\times\overline{X}^B} c_B d(\mu\otimes\nu) &= \int_G \int_{\overline{X}^G} c_B(g,b) d(\mu^{*k}*\nu)(b) d\mu(g)\\ &= \int_G \int_{\overline{X}^G} c_B(g, h\cdot b) d\nu(b) d\mu^{*k}(h) d\mu(g)\\ &= \int_G \int_{\overline{X}^G} c_B(gh,b) - c_B(h,b) d\nu(b) d\mu^{*k}(h) d\mu(g)\\ &= \int_{G\times\overline{X}^B} c_B d(\mu^{*(k+1)}\otimes\nu) - \int_{G\times\overline{X}^B} c_B d(\mu^{*k}\otimes\nu).\end{aligned}$$

Since $\mu^{*0} = \delta_{\mathrm{id}}$,

$$\int_{G\times\overline{X}^B} c_B d(\mu^{*0}\otimes\nu) = \int_{\overline{X}^B} b(o) d\nu(b) = 0$$

and so the above formula is also true for $k = 0$. Then averaging over $k = 0, \ldots, n-1$ and sending $n \to \infty$, we obtain

$$\begin{aligned}\int_{G\times\overline{X}^B} c_B d(\nu\otimes\mu) &= \lim_{n\to\infty} \frac{1}{n} \int_{G\times\overline{X}^B} c_B d(\mu^{*n}\otimes\nu) = \lim_{n\to\infty} \frac{1}{n} \int_{G\times\overline{X}^B} c_B d(\mu^{*n}\otimes\nu)\\ &\leq \lim_{n\to\infty} \frac{1}{n} \int_G \mathrm{d}(o, go) \mu^{*n}(g) = L_\mu,\end{aligned}$$

where in the last equality we used Kingman's ergodic theorem. □

**Lemma 22.5.** *There exists a stationary measure $\nu$ on $\overline{X}^B$ with*

$$\int_{G\times\overline{X}^B} c_B d(\mu\otimes\nu) = L_\mu.$$

*Moreover, if $\mathrm{d}(o, g_1\cdots g_n o) \to +\infty$ almost surely, then $\mathrm{supp}(\nu) \subset \partial_B X$.*

*Proof.* Consider the probability measures $\lambda_k := \int_G \delta_{go} d\mu^{*k}(g)$ and

$$\nu_n := \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k$$

on $\overline{X}^B$. Notice that

$$\mu * \lambda_k = \int_H \int_G h_* \delta_{go} d\mu^{*k}(g) d\mu(h) = \int_H \int_G \delta_{hgo} d\mu^{*k}(g) d\mu(h) = \int_G \delta_{go} d\mu^{*(k+1)}(g) = \lambda_{k+1}.$$

So

$$\mu * \nu_n := \frac{1}{n} \sum_{k=2}^{n} \lambda_k = \nu_n + \frac{1}{n}\lambda_{n+1} - \frac{1}{n}\lambda_1.$$

Hence, any weak-$*$ limit point of $\{\nu_n\}$ is stationary.

Next notice that

$$\begin{aligned}\int_G \int_{\overline{X}^B} c_B(g,b)d\nu_n(b)d\mu(g) &= \int_G \int_{\overline{X}^B} \mathrm{d}(g^{-1}o,b) - \mathrm{d}(o,b)d\nu_n(b)d\mu(g)\\ &= \frac{1}{n}\sum_{k=1}^{n-1} \int_G \int_G \mathrm{d}(g^{-1}o,ho) - \mathrm{d}(o,ho)d\mu^{*k}(h)d\mu(g)\\ &= \frac{1}{n}\sum_{k=1}^{n-1} \int_G \mathrm{d}(o,ho)d\mu^{*(k+1)}(h) - \int_G \mathrm{d}(o,ho)d\mu^{*k}(h)\\ &= \frac{1}{n}\int_G \mathrm{d}(o,ho)d\mu^{*n}(h) - \frac{1}{n}\int_G \mathrm{d}(o,ho)d\mu(h)\end{aligned}$$

which converges to $L_\mu$ by Kingman's ergodic theorem.

Now fix $\nu = \lim_{j\to\infty} \nu_{n_j}$. Then

$$\left|\int_{\overline{X}^B} c_B(g,b)d\nu_n(b)\right| \leq d(go,o)$$

and so by the dominated convergence theorem

$$\begin{aligned}\int_{G\times\overline{X}^B} c_B d(\mu\otimes\nu) &= \int_G \left[\int_{\overline{X}^B} c_B(g,b)d\nu(b)\right] d\mu(g)\\ &= \lim_{j\to\infty} \int_G \int_{\overline{X}^B} c_B(g,b)d\nu_{n_j}(b)d\mu(g) = L_\mu.\end{aligned}$$

Finally assume that $\mathrm{d}(o, g_1\cdots g_n o) \to +\infty$ almost surely. Then for any compact set $K \subset X$, we have

$$\lim_{j\to\infty} \lambda_j(K) = \lim_{j\to\infty} \int_\Omega 1_K(g_1\cdots g_n o)d\beta = 0$$

by the dominated convergence theorem. Thus $\mathrm{supp}(\nu) \subset \partial_B X$. □

To finish the proof we need to use some functional analysis. Consider the function

$$L(\nu) := \int_{G\times\overline{X}^B} c_B d(\mu\otimes\nu)$$

on the space of stationary measures. This is continuous (check!) and affine, i.e. $L(t\nu_1 + (1-t)\nu_2) = tL(\nu_1) + (1-t)L(\nu_2)$. By the above lemmas we know that $L_\mu$ is the maximum value of $L$. Since $L$ is continuous and affine, there exists an extreme point, i.e. an ergodic stationary measure, that realizes the maximum value (this assertion requires Choquet's theorem).

In the case when $\mathrm{d}(o, g_1\cdots g_n o) \to +\infty$ almost surely, we consider $L$ restricted to the space of stationary measures whose support is in $\partial_B X$.

## 23. The Busemann boundary of a Gromov hyperbolic space

In this section we relate the Busemann boundary to the Gromov boundary.

**Theorem 23.1.** *If $X$ is a Gromov hyperbolic metric space, then the identity map $X \to X$ extends to a continuous map*

$$\overline{X}^B \to \overline{X}^{\mathcal{G}}.$$

*Remark* 23.2. In general, the map is not injective on $\partial_B X$.

We start with some general observations.

**Lemma 23.3.** *Suppose $\{x_n\} \subset X$ converges to $b \in \partial_B X$ in $\overline{X}^B$ and to $\xi \in \partial_{\mathcal{G}} X$ in $\overline{X}^{\mathcal{G}}$. For any compact set $K \subset \overline{X}^{\mathcal{G}} - \{\xi\}$ there exists a constant $C = C(K) > 0$ such that: if $x \in K \cap X$, then*

$$b(x) \geq \mathrm{d}(x, o) - C.$$

*Proof.* Fix a neighborhood $U$ of $\xi$ in $\overline{X}^{\mathcal{G}}$ with $K \cap \overline{U} = \emptyset$. By the visibility lemma (Proposition 20.14), there is $R > 0$ so that if $\ell$ is a geodesic joining a point in $U$ to a point in $K$, then

$$\ell \cap B_R(o) \neq \emptyset.$$

Now fix $x \in K \cap X$. For each $n \geq 1$ fix a geodesic $\ell_n$ joining $x$ to $x_n$. Then for $n$ sufficiently large there exists $u_n \in \ell_n$ such that $u_n \in B_R(o)$. Then

$$\begin{aligned} b(x) &= \lim_{n\to\infty} \mathrm{d}(x, x_n) - \mathrm{d}(x_n, o) = \lim_{n\to\infty} \mathrm{d}(x, u_n) + \mathrm{d}(u_n, x_n) - \mathrm{d}(x_n, o) \\ &\geq \lim_{n\to\infty} -2R + \mathrm{d}(x, o) + \mathrm{d}(o, x_n) - \mathrm{d}(x_n, o) \geq -2R + \mathrm{d}(x, o). \end{aligned}$$

□

**Lemma 23.4.** *Suppose $\{x_n\} \subset X$ converges to $b \in \partial_B X$ in $\overline{X}^B$ and to $\xi \in \partial_{\mathcal{G}} X$ in $\overline{X}^{\mathcal{G}}$. If $\ell : [0, \infty) \to X$ is a geodesic ray with $[\ell] = \xi$, then*

$$\lim_{t\to\infty} b(\ell(t)) = -\infty.$$

*Proof.* Let $\ell_n$ be a geodesic joining $o$ to $x_n$. Then passing to a subsequence we can suppose that $\ell_n \to \bar{\ell}$. Then $[\bar{\ell}] = \xi$ and so

$$C := \sup_{t\geq 0} \mathrm{d}(\ell(t), \bar{\ell}(t)) < +\infty.$$

Fix $t \geq 0$. Then

$$\begin{aligned} b(\ell(t)) &= \lim_{n\to\infty} \mathrm{d}(\ell(t), x_n) - \mathrm{d}(o, x_n) = \lim_{n\to\infty} \mathrm{d}(\ell(t), x_n) - \mathrm{d}(o, \ell_n(t)) - \mathrm{d}(\ell_n(t), x_n) \\ &\leq \liminf_{n\to\infty} \mathrm{d}(\ell(t), \ell_n(t)) + \mathrm{d}(\ell_n(t), x_n) - \mathrm{d}(o, \ell_n(t)) - \mathrm{d}(\ell_n(t), x_n) \\ &= \mathrm{d}(\ell(t), \bar{\ell}(t)) - t \leq C - t. \end{aligned}$$

So $b(\ell(t)) \to -\infty$.

□

**Lemma 23.5.** *If $\{x_n\} \subset X$ converges in $\overline{X}^B$, then $\{x_n\}$ converges in $\overline{X}^{\mathcal{G}}$.*

*Proof.* Suppose $x_n \to b$ in $\overline{X}^B$. Suppose $\xi$ is a limit point of $\{x_n\}$ in $\overline{X}^{\mathcal{G}}$. By the previous two lemmas, for every geodesic ray $\ell : [0, \infty) \to X$ we have

$$\lim_{t\to\infty} b(\ell(t)) = \begin{cases} +\infty & \text{if } [\ell] \neq \xi \\ -\infty & \text{if } [\ell] = \xi \end{cases}.$$

This implies that the only limit point of $\{x_n\}$ in $\overline{X}^{\mathcal{G}}$ is $\xi$ and hence $\{x_n\}$ converges in $\overline{X}^{\mathcal{G}}$. □

By the Lemma, the function $F : \overline{X}^B \to \overline{X}^{\mathcal{G}}$ given by

$$F(b) = \begin{cases} b & \text{if } b \in X \\ \lim_{X \ni x \to b} x & \text{if } b \in \partial_B X \end{cases}$$

is well defined. Using the Lemma, one can check that $F$ is continuous.

## 24. Positive Linear Drift

Suppose $X$ is a Gromov hyperbolic space, and $\mu$ is a probability measure on $\mathsf{Isom}(X)$ where

$$\int_{\mathsf{Isom}(X)} \mathrm{d}(o, go) d\mu(g) < +\infty.$$

and the semigroup generated by $\mathrm{supp}(\mu)$ is non-elementary. We finish the proof of Theorem 19.1 by proving that the linear drift is positive.

Fix $o \in X$ and let $\overline{X}^B$ be the associated Busemann compactification. Define $f : \Omega \times \partial_B X \to \mathbb{R}$ by $f(\mathbf{g}, b) = c_B(g_1, b) = b(g_1^{-1} o)$. By Theorem 21.1, we know that $\mathrm{d}(o, g_1 \cdots g_n o) \to +\infty$ almost surely. So by Theorem 22.3, there exists an ergodic stationary measure $\nu$ on $\partial_B X$ such that

$$L_\mu = \int_{\mathsf{Isom}(X) \times \partial_B X} c_B(g, b) d(\mu \otimes \nu)(g, b) = \int_{\Omega \times \partial_B X} f d(\beta \otimes \nu).$$

Let $S : \Omega \times \partial_B X \to \Omega \times \partial_B X$ be the extended shift map $S(\mathbf{g}, b) = (T\mathbf{g}, g_1 \cdot b)$. Since $\nu$ is ergodic, Proposition 8.5 implies that $S$ is ergodic with respect to $\beta \otimes \nu$. Then, using Proposition 15.1, to prove that $L_\mu > 0$ it suffices to show that

$$\sum_{j=0}^{n-1} f \circ S^j(\mathbf{g}, b) \to +\infty$$

almost everywhere. Recall that

$$g \cdot b = b \circ g^{-1} - b(g^{-1} o)$$

and so

$$\begin{aligned}
\sum_{j=0}^{n-1} f \circ S^j(\mathbf{g}, b) &= \sum_{j=0}^{n-1} f(T^j(\mathbf{g}), g_j \cdots g_1 b) = \sum_{j=0}^{n-1} g_j \cdots g_1 b(g_{j+1}^{-1} o) \\
&= \sum_{j=0}^{n-1} b(g_1^{-1} \cdots g_{j+1}^{-1} o) - b(g_1^{-1} \cdots g_j^{-1} o) \\
&= b(g_1^{-1} \cdots g_n^{-1} o) - b(o) = b(g_1^{-1} \cdots g_n^{-1} o).
\end{aligned}$$

**Lemma 24.1.** *There is a Borel map $\hat{\zeta} : \Omega \to \partial_{\mathcal{G}} X$ such that for $\beta$-almost every $\mathbf{g}$ we have*

$$\lim_{n \to \infty} g_1^{-1} \cdots g_n^{-1}(o) = \hat{\zeta}(\mathbf{g}).$$

*Proof.* Let $i : \mathsf{Isom}(X) \to \mathsf{Isom}(X)$ be inversion $i(g) = g^{-1}$ and let $\hat{\mu} := i_* \mu$. The semigroup generated by $\mathrm{supp}(\hat{\mu})$ is non-elementary (check!) and so by Theorem 21.1 there is a Borel map $\hat{\zeta}_0 : \Omega \to \partial_{\mathcal{G}} X$ such that for $\hat{\beta} := \hat{\mu}^{\otimes \mathbb{N}}$-almost every $\mathbf{h}$ we have

$$\lim_{n \to \infty} h_1 \cdots h_n(o) = \hat{\zeta}_0(\mathbf{h}).$$

Equivalently, for $\beta$-almost every $\mathbf{g}$ we have

$$\lim_{n \to \infty} g_1^{-1} \cdots g_n^{-1}(o) = \hat{\zeta}(\mathbf{g}).$$

where $\hat{\zeta}(g_1, g_2, \dots) := \hat{\zeta}_0(g_1^{-1}, g_2^{-1}, \dots)$. □

Let $\pi : \partial_B X \to \partial_{\mathcal{G}} X$ be the continuous map from Theorem 23.1. Then $\pi_*\nu$ is a stationary measure on $\partial_{\mathcal{G}} X$. So by Theorem 21.1 $\pi_*\nu$ is the unique stationary measure on $\partial_{\mathcal{G}} X$ and has no atoms. Thus for any $\xi \in \partial_{\mathcal{G}} X$,

$$\nu(\pi^{-1}(\xi)) = \pi_*\nu(\{\xi\}) = 0.$$

Then

$$(\beta \otimes \nu)\Big(\{(\mathbf{g}, b) : \lim_{n\to\infty} g_1^{-1} \cdots g_n^{-1}(o) = \hat{\zeta}(\mathbf{g}) \text{ and } \pi(b) \neq \hat{\zeta}(\mathbf{g})\}\Big)$$
$$= \int_\Omega 1 - \nu\big(\pi^{-1}(\hat{\zeta}(\mathbf{g}))\big) d\beta(\mathbf{g}) = \int_\Omega 1 d\beta = 1.$$

For $(\mathbf{g}, b)$ in this full measure set, Lemma 23.3 implies that there exists a constant $C = C(\mathbf{g}, b)$ such that

$$b(g_1^{-1} \cdots g_n^{-1} o) > \mathrm{d}(g_1^{-1} \cdots g_n^{-1} o, o) - C$$

and hence $b(g_1^{-1} \cdots g_n^{-1} o) \to +\infty$.

So

$$\sum_{j=0}^{n-1} f \circ S^j(\mathbf{g}, b) = b(g_1^{-1} \cdots g_n^{-1} o) \to +\infty$$

almost everywhere. Thus by Proposition 15.1,

$$L_\mu = \int_{\Omega \times \partial_B X} f d(\beta \otimes \nu) > 0.$$

# Part 4. Random walks on finitely generated groups

## 25. Introduction

In this part of the lecture notes, we consider the case when $\Gamma$ is a finitely generated group (we now use $\Gamma$ to denote our group).

Given a probability measure $\mu$ on $\Gamma$ and $g_1, g_2, \ldots$ independent identically distributed elements of $\Gamma$ each with distribution $\mu$, we can view the random products

$$X_n = g_1 \cdots g_n$$

as a random walk on $\Gamma$ starting at id. Notice that the distribution of $X_{n-1}^{-1} X_n$ is $\mu$ and so $\mu$ is called the *step distribution* of the random walk. Also notice that

$$p_n(\alpha, \beta) := \mu^{*n}(\alpha^{-1}\beta).$$

represents the probability that a random walk moves from $\alpha$ to $\beta$ after $n$-steps.

Finally, we can view $\Omega = \Gamma^{\mathbb{N}}$ as the space of all possible steps along random walks and $\beta = \mu^{\otimes \mathbb{N}}$ as the probability that a set of steps occur.

Now we introduce the terminology we will use through out this part of the lecture notes.

**Word length.** Given a finite generating set $S \subset \Gamma$ (i.e. every element of $\Gamma$ can be written as a product of elements in $S$), we let

$$|\gamma|_S := \min\{k : \gamma \in S^k\}$$

denote the *word length with respect to* $S$. If $S' \subset \Gamma$ is another finite generating set, then there exists $C > 1$ such that

$$\frac{1}{C} |\gamma|_S \leq |\gamma|_{S'} \leq C |\gamma|_S$$

for all $\gamma \in \Gamma$. So in many cases, the exact choice of finite generating set doesn't matter.

In the case when $S$ is symmetric, i.e. $s \in S \Leftrightarrow s^{-1} \in S$, then the function

$$\mathrm{d}_S(g,h) = \left|g^{-1}h\right|_S$$

is a metric on $\Gamma$ called the *word metric induced by* $S$.

**Symmetric.** A random walk is *symmetric* if

$$\mu(g) = \mu(g^{-1})$$

for all $g \in \Gamma$.

**Irreducible.** A random walk is *irreducible* if $\Gamma$ is generated by $\mathrm{supp}(\mu)$ as a semigroup, i.e.

$$\Gamma = \bigcup_{n\geq 1} \mathrm{supp}(\mu^{*n}) = \bigcup_{n\geq 1} \mathrm{supp}(\mu)^n.$$

**Recurrent and Transient.** A random walk is *recurrent* if

$$\Pr(X_n = \mathrm{id} \text{ for some } n \geq 1) = 1,$$

i.e. if the random walk almost surely returns to id, otherwise it is *transient*.

**Observation 25.1.** If $F_n \subset \Omega$ is the set of random walks that return to id at least $n$ times, then $\beta(F_n) = \beta(F_1)^n$.

*Proof.* Fix $n > 1$. For $m \in \mathbb{N}$, let $F_{n-1,m} \subset \Omega$ be the set of random walks that return to id at step $m$ and return to id exactly $n-1$ times in the first $m$-steps. Notice that

$$F_{n-1} = \bigsqcup_{m\geq n-1} F_{n-1,m} \quad \text{and} \quad F_n = \bigsqcup_{m\geq n-1} \pi_m(F_{n-1,m}) \times F_1$$

where $\pi_m : \Omega \to \Gamma^m$ is the projection onto the first $m$-coordinates. Also notice that $F_{n-1,m} = \pi_m(F_{n-1,m}) \times \Gamma^{\mathbb{N}}$ and so

$$\beta(F_{n-1}) = \sum_{m\geq n-1} \beta(F_{n-1,m}) = \sum_{m\geq 1} \mu^{\otimes m}(\pi_m(F_{n-1,m}))$$

and

$$\begin{aligned}\beta(F_n) &= \sum_{m\geq n-1} \beta(\pi_m(F_{n-1,m}) \times F_1) = \sum_{m\geq 1} \mu^{\otimes m}(\pi_m(F_{n-1,m}))\beta(F_1) \\ &= \beta(F_{n-1})\beta(F_1).\end{aligned}$$

Then by induction $\beta(F_n) = \beta(F_1)^n$.

□

**Proposition 25.2.** *Given a random walk, the following are equivalent:*

*(1) The random walk is recurrent.*
*(2)* $\Pr(X_n = \mathrm{id}$ *for infinitely many* $n \geq 1) = 1$.
*(3)* $\sum_{n=1}^{\infty} p_n(\mathrm{id}, \mathrm{id}) = +\infty$.

*If the random walk is irreducible, the above are also equivalent to:*

*(4)* $\sum_{n=1}^{\infty} p_n(\alpha, \beta) = +\infty$ *for some* $\alpha, \beta \in \Gamma$.
*(5)* $\sum_{n=1}^{\infty} p_n(\alpha, \beta) = +\infty$ *for all* $\alpha, \beta \in \Gamma$.

*Proof.* (2) $\Rightarrow$ (1) is by definition. If (1) is true, then $\beta(F_n) = \beta(F_1)^n = 1$ and so

$$\Pr(X_n = \mathrm{id} \text{ for infinitely many } n \geq 1) = \beta(\cap_{n\geq 1} F_n) = 1.$$

Thus (1) $\Rightarrow$ (2).

Notice that

$$\sum_{n=1}^{\infty} 1_{\{g_1\cdots g_n = \mathrm{id}\}}(\mathbf{g}) = \begin{array}{c}\#\text{ of times the random walk} \\ \text{corresponding to } \mathbf{g} \text{ returns to id}\end{array} = \sum_{n=1}^{\infty} 1_{F_n}(\mathbf{g}).$$

So

$$\begin{aligned}\sum_{n=1}^{\infty} p_n(\mathrm{id},\mathrm{id}) = \sum_{n=1}^{\infty} \beta(\{g_1\cdots g_n = \mathrm{id}\}) = \sum_{n=1}^{\infty}\int 1_{\{g_1\cdots g_n=\mathrm{id}\}} d\beta = \sum_{n=1}^{\infty}\int 1_{F_n} d\beta \\ = \sum_{n=1}^{\infty}\beta(F_n) = \sum_{n=1}^{\infty}\beta(F_1)^n.\end{aligned}$$

Thus (1) $\Leftrightarrow$ (3).

Now suppose that the random walk is irreducible. Clearly (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (3). So it suffices to show (4) $\Rightarrow$ (5). Suppose $\sum_{n=1}^{\infty} p_n(\alpha_0,\beta_0) = +\infty$ and fix $\alpha,\beta \in \Gamma$. By irreducibility, there exist $k,\ell \geq 1$ such that $p_k(\alpha,\alpha_0) > 0$ and $p_\ell(\beta_0,\beta) > 0$. Then

$$p_n(\alpha,\beta) \geq p_k(\alpha,\alpha_0)p_{n-k-\ell}(\alpha_0,\beta_0)p_\ell(\beta_0,\beta)$$

for $n > k+\ell$ and so

$$\sum_{n=1}^{\infty} p_n(\alpha,\beta) \geq p_k(\alpha,\alpha_0)p_\ell(\beta_0,\beta)\sum_{n=k+\ell+1}^{\infty} p_n(\alpha_0,\beta_0) = +\infty.$$

□

We state two major theorems without proof (proofs of both can be found in [Lal23]).

**Theorem 25.3** (Theorem 7.2.2 in [Lal23])**.** *If there is an irreducible recurrent symmetric random walk on* $\Gamma$*, then every symmetric random walk with finite support is also recurrent.*

**Definition 25.4.** An infinite finitely generated group is *recurrent* if some irreducible symmetric random walk on the group is recurrent, otherwise the group is *transient*.

**Theorem 25.5** (Varopoulos' Recurrence Criterion (Theorem 7.3.1 in [Lal23]))**.** *An infinite finitely generated group is recurrent if and only if it has a finite-index subgroup isomorphic to either* $\mathbb{Z}$ *or* $\mathbb{Z}^2$*.*

We will almost always consider transient irreducible random walks.

## 26. Amenability and linear drift

In this section we introduce amenable groups and prove positivity of linear drift for certain random walks on non-amenable groups.

**Theorem 26.1.** *Suppose* $\Gamma$ *is a finitely generated group,* $\mathrm{d}_S$ *is a word metric on* $\Gamma$*, and* $\mu$ *is a probability measure which induces an irreducible random walk. If* $\Gamma$ *is non-amenable, then the linear drift*

$$L_\mu := \lim_{n\to\infty} \frac{\mathrm{d}_S(\mathrm{id}, X_n)}{n}$$

*is positive (possibly equal to $+\infty$).*

In Section 26.1, we will introduce amenable groups and prove some basic properties. In Section 26.2, we prove several characterizations of amenability. In Section 26.3, we introduce the spectral radius of a random walk and prove that if it is less than one, then the linear drift is positive. Finally, in Section 26.4 we relate the spectral radius to amenability.

26.1. **Amenable groups.** Given a countable group $\Gamma$, a *Følner sequence* is a sequence $\{F_n\} \subset \Gamma$ of finite sets such that

$$\frac{|F_n \Delta g F_n|}{|F_n|} \to 0$$

for all $g \in \Gamma$. A countable group $\Gamma$ is *amenable* if it admits a Følner sequence.

**Example 26.2.** $(\mathbb{Z}, +)$ is amenable: If $F_n = \{0, 1, \dots, n\}$, then

$$\frac{|F_n \Delta (m + F_n)|}{|F_n|} = \frac{2\,|m|}{n} \to 0$$

for all $m \in \mathbb{Z}$. More generally, a solvable group is amenable.

Amenable groups have the following important property.

**Proposition 26.3.** *If $\Gamma$ is amenable and acts continuously on a compact metrizable space $Y$, then there exists a $\Gamma$-invariant Borel probability measure on $Y$.*

*Proof.* Fix a Følner sequence $\{F_n\}$ and fix $y_0 \in Y$. Define a Borel probability measure

$$\mu_n := \frac{1}{|F_n|} \sum_{g \in F_n} \delta_{g y_0}.$$

Notice that if $h \in \Gamma$, then

$$\mu_n - h_* \mu_n = \frac{1}{|F_n|} \left( \sum_{g \in F_n \setminus h F_n} \delta_{g y_0} - \sum_{g \in h F_n \setminus F_n} \delta_{g y_0} \right).$$

So if $f : Y \to \mathbb{R}$ is continuous, then

$$\left| \int f d\mu_n - \int f d(h_* \mu_n) \right| \leq \|f\|_\infty \frac{|F_n \Delta h F_n|}{|F_n|} \to 0.$$

So any weak-$*$ limit of $\{\mu_n\}$ is $\Gamma$-invariant. □

**Example 26.4.** If $\Gamma$ is a countable group which is either isomorphic to

(1) a Zariski dense subgroup of $\mathsf{SL}(d, \mathbb{R})$,
(2) a non-elementary subgroup of the isometry group of a Gromov hyperbolic space,

then $\Gamma$ is not amenable.

*Proof.* We just prove (2). We will use our previous results about stationary measures, but one could provide a much more direct argument.

Suppose $\Gamma \subset \mathsf{Isom}(X)$ is non-elementary where $X$ is Gromov hyperbolic. Suppose for a contradiction that $\Gamma$ is amenable. Then there exists a $\Gamma$-invariant Borel

probability measure $\nu$ on the Gromov boundary $\partial_G X$. Now fix a probability measure $\mu$ on $\Gamma$ with full support. Then

$$\mu * \nu = \sum \mu(g) g_* \nu = \sum \mu(g) \nu = \nu$$

and so $\nu$ is stationary. By Theorem 21.1, for a generic $\mathbf{g} \in \Gamma^{\mathbb{N}}$ we have

$$\lim_{n\to\infty} (g_1 \cdots g_n)_* \nu = \delta_{\zeta(\mathbf{g})}$$

for some $\zeta(\mathbf{g}) \in \partial_G X$. Then by $\Gamma$-invariance, $\nu = \delta_{\zeta(\mathbf{g})}$. However, Theorem 21.1 also says that $\nu$ has no atoms. So we have a contradiction. □

**Proposition 26.5.** *If $\Gamma$ is amenable and $H \subset \Gamma$ is a subgroup, then $H$ is amenable.*

*Proof.* We claim that for any finite set $S \subset H$ and any $\epsilon > 0$ there exists a finite set $F \subset H$ such that

$$\max_{h\in S} \frac{|F \Delta hF|}{|F|} < \epsilon.$$

Since $\Gamma$ is amenable, there exists a finite set $E \subset \Gamma$ such that

$$\max_{h\in S} \frac{|E \Delta hE|}{|E|} < \frac{\epsilon}{|S|}.$$

Fix a set of coset representations $R$ of $H\backslash\Gamma$. For $r \in R$, let $E^r := E \cap (Hr)$. Notice that

$$\sum_{h\in S} \sum_{r\in R} |E^r \Delta hE^r| = \sum_{h\in S} |E \Delta hE| < \epsilon\, |E| = \epsilon \sum_{r\in R} |E^r|\,.$$

So there exists some $r \in R$ where $\sum_{h\in S} |E^r \Delta hE^r| < \epsilon\, |E^r|$. For such a set, let $F := E^r r^{-1} \subset H$. Then

$$\max_{h\in S} \frac{|F \Delta hF|}{|F|} = \max_{h\in S} \frac{|E^r \Delta hE^r|}{|E^r|} < \epsilon.$$

Now fix finite sets $S_1 \subset S_2 \subset \cdots$ such that $H = \cup S_n$ and let $\{F_n\} \subset H$ be finite sets with

$$\max_{h\in S_n} \frac{|F_n \Delta hF_n|}{|F_n|} < \frac{1}{n}.$$

Then $\{F_n\}$ is a Følner sequence. □

**Corollary 26.6.** *If $\Gamma$ contains a free group on two generators, then $\Gamma$ is not amenable.*

*Proof.* Suppose $H = \langle a, b\rangle$ is a free subgroup. Then let $X$ be the Cayley graph of $H$ associated to the generators $\{a, b, a^{-1}, b^{-1}\}$. Then $X$ is Gromov hyperbolic and the action of $H$ on $X$ is non-elementary. Hence $H$ is non-amenable. So $\Gamma$ is non-amenable.

□

26.2. **Equivalent definitions.** We now describe several characterizations of amenability. For the rest of the section suppose that $\Gamma$ is a countable group.

We will consider two actions of $\Gamma$ on the set of functions $\Gamma \to \mathbb{R}$. The *right action* is

$$R_g f(h) = f(hg)$$

and the *left action* is

$$L_g f(h) = f(g^{-1}h).$$

For $p \in [1, \infty]$ let $L^p(\Gamma)$ be the usual $L^p$-space of functions $\Gamma \mapsto \mathbb{R}$ associated to the counting measure. Notice that both the right and the left actions are by isometries on every $L^p(\Gamma)$.

A *mean* on $\Gamma$ is a continuous functional $\ell \in L^\infty(\Gamma)^*$ where

(1) $\ell(f) \geq 0$ for all $f \geq 0$,
(2) $\ell(1_\Gamma) = 1$.

A mean $\ell$ is $\Gamma$-invariant if $\ell(f) = \ell(L_g f)$ for all $f \in L^\infty(\Gamma)$ and $g \in G$.

*Remark* 26.7. Given a $\Gamma$-invariant mean $\ell \in L^\infty(\Gamma)^*$, the function

$$A \mapsto \ell(1_A)$$

defines a $\Gamma$-invariant finitely additive probability measure on $\Gamma$. Conversely given a $\Gamma$-invariant finitely additive probability measure $\mu$ on $\Gamma$, the function

$$f \mapsto \int f d\mu$$

defines a $\Gamma$-invariant mean on $\Gamma$.

**Theorem 26.8.** *The following are equivalent:*

*(1)* $\Gamma$ *is amenable.*
*(2)* $\Gamma$ *admits a mean (i.e. it is "a mean-able").*
*(3)* $\Gamma$ *admits a* $\Gamma$*-invariant finitely additive probability measure.*
*(4) There exists a sequence* $\{f_n\} \subset L^1(\Gamma)$ *of unit vectors where* $\|f_n - L_g f_n\|_1 \to 0$ *for all* $g \in \Gamma$.
*(5) There exists a sequence* $\{f_n\} \subset L^1(\Gamma)$ *of unit vectors where* $\|f_n - R_g f_n\|_1 \to 0$ *for all* $g \in \Gamma$.
*(6) There exists a sequence* $\{f_n\} \subset L^2(\Gamma)$ *of unit vectors where* $\|f_n - L_g f_n\|_2 \to 0$ *for all* $g \in \Gamma$.
*(7) There exists a sequence* $\{f_n\} \subset L^2(\Gamma)$ *of unit vectors where* $\|f_n - R_g f_n\|_2 \to 0$ *for all* $g \in \Gamma$.

*Proof.* We start with the easy directions. Recall that (2) $\Leftrightarrow$ (3) by the discussion above.

Given $f : \Gamma \to \mathbb{R}$, define $\Phi(f) : \Gamma \to \mathbb{R}$ by $\Phi(f)(g) = f(g^{-1})$. Then $\Phi$ induces an isometry of every $L^p(\Gamma)$ and

$$\Phi \circ L_g = R_g \circ \Phi.$$

Using $\Phi$ one can show that (4) $\Leftrightarrow$ (5) and (6) $\Leftrightarrow$ (7).

(4) $\Rightarrow$ (6). Suppose $\{f_n\} \subset L^1(\Gamma)$ are unit vectors and $\|f_n - L_g f_n\|_1 \to 0$ for all $g \in \Gamma$. Let $h_n := |f_n|^{1/2}$. Then

$$\begin{aligned}\|h_n - L_g h_n\|_2^2 &= \sum_{\gamma\in\Gamma} \left||f_n(\gamma)|^{1/2} - \left|f_n(g^{-1}\gamma)\right|^{1/2}\right|^2 \leq \sum_{\gamma\in\Gamma} \left||f_n(\gamma)| - \left|f_n(g^{-1}\gamma)\right|\right| \\ &\leq \|f_n - L_g f_n\|_1 \to 0.\end{aligned}$$

(6) $\Rightarrow$ (4). Suppose $\{f_n\} \subset L^2(\Gamma)$ are unit vectors and $\|f_n - L_g f_n\|_2 \to 0$ for all $g \in \Gamma$. Let $h_n := f_n^2$. Then

$$\begin{aligned}\|h_n - L_g h_n\|_1 &= \|(f_n + L_g f_n)(f_n - L_g f_n)\|_1 \leq \|f_n + L_g f_n\|_2 \|f_n - L_g f_n\|_2 \\ &\leq 2\|f_n - L_g f_n\|_2 \to 0.\end{aligned}$$

(1) $\Rightarrow$ (3). Let $\mathcal{M}$ denote the set of finitely additive probability measures on $\Gamma$, which we view as a compact subset of $[0,1]^{\mathcal{P}(\Gamma)}$.

We claim that for any finite set $S \subset \Gamma$ and any $\epsilon > 0$ the set

$$\mathcal{M}(S,\epsilon) := \{\mu : |\mu(E) - \mu(gE)| \leq \epsilon \text{ for all } E \in \mathcal{P}(\Gamma) \text{ and } g \in S\}$$

is non-empty. Since $\Gamma$ is amenable, there exists a finite set $F \subset \Gamma$ so that

$$\max_{g\in S} \frac{|F\Delta gF|}{|F|} \leq \epsilon.$$

Define $\mu \in \mathcal{M}$ by

$$\mu(A) = \frac{1}{|F|}|A \cap F|.$$

Then

$$||gA \cap F| - |A \cap F|| = \left||A \cap g^{-1}F| - |A \cap F|\right| \leq \left|F\Delta g^{-1}F\right| = |(gF)\Delta F|$$

and so $\mu \in \mathcal{M}(S,\epsilon)$.

Now fix finite sets $S_1 \subset S_2 \subset \cdots$ such that $\Gamma = \cup S_n$. Then by the nested intersection property

$$\bigcap_{n=1}^{\infty} \mathcal{M}(S_n, 1/n)$$

is non-empty and any element of this set is a $\Gamma$-invariant finitely additive probability measure.

(2) $\Rightarrow$ (4). We claim that for any finite set $S \subset \Gamma$ and any $\epsilon > 0$ there exists a unit vector $f \in L^1(\Gamma)$ such that

$$\max_{g\in S} \|f - L_g f\|_1 < \epsilon.$$

Suppose not. Consider the set

$$\{(f - L_g f)_{g\in S} : f \geq 0 \text{ and } \|f\|_1 = 1\} \subset L^1(\Gamma)^S$$

This is convex and bounded away from zero. Since $(L^1(\Gamma)^S)^* = L^\infty(\Gamma)^S$, by the Hahn-Banach Separation Theorem there exist $\delta > 0$ and $(\rho_g)_{g\in S} \in L^\infty(\Gamma)^S$ such that

$$\sum_{g\in S} \langle \rho_g, f - L_g f\rangle \geq \delta$$

when $f \geq 0$ and $\|f\|_1 = 1$. Notice that

$$\langle \rho_g, L_g f\rangle = \sum_{h\in\Gamma} \rho_g(h) f(g^{-1}h) = \sum_{h\in\Gamma} \rho_g(gh) f(h) = \left\langle L_{g^{-1}}\rho_g, f\right\rangle.$$

So

$$\left\langle f, \sum_{g\in S} \rho_g - L_{g^{-1}}\rho_g \right\rangle \geq \delta \|f\|_1 \tag{6}$$

whenever $f \in L^1(\Gamma)$ and $f \geq 0$.

Now let $\rho := \sum_{g\in S} \rho_g - L_{g^{-1}}\rho_g$. Plugging $f = \delta_x$ into Equation (6), we see that $\rho(x) \geq \delta$. Hence $\rho \geq \delta 1_\Gamma$. Now fix a $\Gamma$-invariant mean $\ell : L^\infty(\Gamma) \to \mathbb{R}$. Then $\ell(\rho) \geq \delta\ell(1_\Gamma) = \delta$, but by $\Gamma$-invariance

$$\ell(\rho) = \ell\left(\sum_{g\in S} \rho_g - L_{g^{-1}}\rho_g\right) = 0.$$

Contradiction!

Now fix finite sets $S_1 \subset S_2 \subset \cdots$ such that $\Gamma = \cup S_n$. For each $n$ there exists a unit vector $f_n \in L^1(\Gamma)$ with

$$\max_{g\in S_n} \|f_n - L_g f_n\|_1 < \frac{1}{n}.$$

Then $\{f_n\}$ is our desired sequence.

(4) $\Rightarrow$ (1). We claim that for any finite set $S \subset \Gamma$ and any $\epsilon > 0$ there exists a finite set $F \subset \Gamma$ such that

$$\max_{g\in S} \frac{|F\Delta gF|}{|F|} < \epsilon.$$

By (4), we can fix a unit vector $f \in L^1(\Gamma)$ where

$$\max_{g\in S} \|f - L_g f\|_1 < \frac{\epsilon}{|S|}.$$

By approximation, we can suppose that $f$ is finitely supported. By replacing $f$ with $|f|$, we can also assume that $f \geq 0$. Then we can fix $A_1 \supset \cdots \supset A_k$ such that

$$f_n = \sum c_j 1_{A_j}.$$

and each $c_j$ is positive. Notice that

$$f - L_g f_n = \sum c_j(1_{A_j} - 1_{gA_j}) = \sum c_j(1_{A_j \smallsetminus gA_j} - 1_{gA_j \smallsetminus A_j}).$$

Since the sets $(A_j)$ are nested,

$$(A_j \smallsetminus gA_j) \cap (gA_\ell \smallsetminus A_\ell) = \emptyset$$

for all $1 \leq j, \ell \leq k$. So no cancellation occurs. Hence

$$\|f - L_g f\|_1 = \sum_j c_j \left|A_j \Delta gA_j\right|.$$

Also $1 = \|f\|_1 = \sum c_j |A_j|$. So

$$\sum_{g\in S}\sum_j c_j \left|A_j \Delta gA_j\right| = \sum_{g\in S} \|f - L_g f\|_1 \leq \epsilon = \epsilon \sum_j c_j \left|A_j\right|.$$

So there exists $j$ where

$$\sum_{g\in S} \left|A_j \Delta gA_j\right| \leq \epsilon \left|A_j\right|.$$

Then $F = A_j$ satisfies the claim.

Now fix finite sets $S_1 \subset S_2 \subset \cdots$ such that $\Gamma = \cup S_n$. For each $n$ there exists a finite set $F_n \subset \Gamma$ with

$$\max_{g \in S_n} \frac{|F_n \Delta g F_n|}{|F_n|} < \frac{1}{n}.$$

Then $\{F_n\}$ is a Følner sequence.

□

26.3. **Spectral radius of a random walk.** Suppose $\Gamma$ is a countable group and $\mu$ is a probability measure on $\Gamma$ so that the associated random walk is irreducible. The *spectral radius* of $\mu$ is

$$\rho(\mu) := \limsup_{n \to \infty} \mu^{*n}(\mathrm{id})^{1/n}.$$

**Observation 26.9.** If $g \in \Gamma$, then

$$\rho(\mu) = \limsup_{n \to \infty} \mu^{*n}(g)^{1/n}.$$

*Proof.* Fix $k, l$ such that $\mu^{*k}(g) > 0$ and $\mu^{*l}(g^{-1}) > 0$. Then

$$\mu^{*n}(g) \geq \mu^{*(n-k)}(\mathrm{id})\mu^{*k}(g)$$

and

$$\mu^{*n}(\mathrm{id}) \leq \mu^{*(n-l)}(g)\mu^{*l}(g^{-1}).$$

So

$$\limsup_{n \to \infty} \mu^{*n}(g)^{1/n} = \limsup_{n \to \infty} \mu^{*n}(\mathrm{id})^{1/n}.$$

□

We relate the spectral radius to the linear drift.

**Theorem 26.10.** *Suppose $\mathrm{d}_S$ is a word metric on $\Gamma$. If $\rho(\mu) < 1$, then the linear drift with respect to $\mathrm{d}_S$ is positive.*

**Lemma 26.11.** *For any $\rho(\mu) < \rho_0$ there exist $C, a > 0$ such that*

$$\mu^{*n}(g) \leq C a^{|g|_S} \rho_0^n.$$

*Proof.* Fix $C_1 > 0$ such that

$$\mu^{*n}(\mathrm{id}) \leq C_1 \rho_0^n.$$

For each $s \in S$ fix $n(s) \in \mathbb{N}$ such that $\mu^{*n(s)}(s) > 0$. Then let $\epsilon := \min_{s \in S} \mu^{*n(s)}(s)$ and $N := \max_{s \in S} n(s)$.

Fix $g \in \Gamma$. Then $g^{-1} = s_1 \cdots s_k$ where $s_1, \ldots, s_k \in S$ and $k = \left|g^{-1}\right|_S = |g|_S$. Let $k' = n(s_1) + \cdots + n(s_k)$. Then

$$\mu^{*k'}(g^{-1}) \geq \epsilon^k \geq \epsilon^{k'}.$$

So

$$C_1 \rho_0^{n+k'} \geq \mu^{*(n+k')}(\mathrm{id}) \geq \mu^{*n}(g)\mu^{*k'}(g^{-1}) \geq \mu^{*n}(g)\epsilon^{k'}$$

Then

$$\mu^{*n}(g) \leq C_1 (\rho_0/\epsilon)^{k'} \rho_0^n \leq C_1 a^{|g|_S} \rho_0^n$$

where $a := (\rho_0/\epsilon)^N$.

□

*Proof of Theorem 26.10.* Fix $\rho(\mu) < \rho_0 < 1$ and let $C, a > 0$ be as in the lemma. Then fix $\delta > 0$ such that

$$|S|^\delta a^\delta \rho_0 < 1.$$

Notice that

$$|\{g : |g|_S \le \delta n\}| \le |S|^{\delta n}.$$

So

$$\begin{aligned}\Pr(|X_n| \le \delta n) &= \sum_{|g|_S \le \delta n} \mu^{*n}(g) \le |S|^{\delta n} C a^{\delta n} \rho_0^n \\ &= C\left(|S|^\delta a^\delta \rho_0\right)^n \to 0.\end{aligned}$$

Thus $L_\mu \ge \delta$. □

26.4. **Spectral radius and amenability.** Suppose $\Gamma$ is a countable group and $\mu$ is a probability measure on $\Gamma$ so that the associated random walk is irreducible.

**Theorem 26.12** (Kesten, Day)**.**

*(1) If $\rho(\mu) = 1$, then $\Gamma$ is amenable.*
*(2) If $\mu$ is symmetric and $\Gamma$ is amenable, then $\rho(\mu) = 1$.*

*Remark* 26.13. This theorem is due to Kesten [Kes59b, Kes59a] in the symmetric case and to Day [Day64] in the non-symmetric case.

The rest of the section is devoted to the proof of the theorem. Define $P_\mu : L^2(\Gamma) \to L^2(\Gamma)$ by

$$P_\mu f(x) = \sum_{g\in\Gamma} \mu(g) f(xg).$$

Notice that $P_\mu^k = P_{\mu^{*k}}$.

Recall that we previously defined an isometric action of $\Gamma$ on $L^2(\Gamma)$ given by

$$R_g f(x) = f(xg).$$

Then we can write

$$P_\mu = \sum_{g\in\Gamma} \mu(g) R_g.$$

**Lemma 26.14.** $\sup_{g\in\Gamma} \mu^{*k}(g) \le \left\|P_\mu^k\right\| \le 1$ *for all* $k \ge 1$.

*Proof.* Notice that

$$\left\|P_\mu^k f\right\|_2 = \left\|\sum_{g\in\Gamma} \mu^{*k}(g) R_g f\right\| \le \sum_{g\in\Gamma} \mu^{*k}(g) \left\|R_g f\right\|_2 = \|f\|_2.$$

Hence $\left\|P_\mu^k\right\| \le 1$. Further,

$$P_\mu^k \delta_{\mathrm{id}}(g) = \sum_{\gamma\in\Gamma} \mu^{*k}(\gamma) \delta_{\mathrm{id}}(g\gamma) = \mu^{*k}(g^{-1})$$

and so

$$\left\langle P_\mu^k \delta_{\mathrm{id}}, \delta_{g^{-1}} \right\rangle = \mu^{*k}(g).$$

Since $\delta_{\mathrm{id}}, \delta_{g^{-1}}$ are unit vectors in $L^2(\Gamma)$, we must have $\left\|P_\mu^k\right\| \ge \mu^{*k}(g)$.

□

**Lemma 26.15.** *$\rho(\mu)^k \le \left\|P_\mu^k\right\|$ for all $k$. Moreover, if $\mu$ is symmetric, then $\left\|P_\mu^k\right\| = \rho(\mu)^k$ for all $k$.*

*Proof.* Fix $k \ge 1$. Then any $n$ can be written as $n = a_n k + b_n$ where $b_n \in \{0, 1, \dots, k-1\}$. Then

$$\left\|P_\mu^n\right\| \le \left\|P_\mu^k\right\|^{a_n} \left\|P_\mu\right\|^{b_n}.$$

Since $a_n/n \to 1/k$ and $\left\|P_\mu^n\right\| \ge \mu^{*n}(\mathrm{id})$,

$$\begin{aligned}\left\|P_\mu^k\right\|^{1/k} &= \lim_{n\to\infty} \left\|P_\mu^k\right\|^{a_n/n} \left\|P_\mu\right\|^{b_n/n} \ge \limsup_{n\to\infty} \left\|P_\mu^n\right\|^{1/n} \\ &\ge \limsup_{n\to\infty} \mu^{*n}(\mathrm{id})^{1/n} = \rho(\mu).\end{aligned}$$

Now suppose $\mu$ is symmetric. We first claim that $P_\mu$ is a symmetric operator. Notice that

$$\langle R_g f, h\rangle = \sum_x f(xg)h(x) = \sum_y f(y)h(yg^{-1}) = \left\langle f, R_{g^{-1}} h\right\rangle.$$

Hence $R_g^t = R_{g^{-1}}$. Thus

$$P_\mu^t = \sum \mu(g) R_g^t = \sum \mu(g) R_{g^{-1}} = \sum \mu(g^{-1}) R_{g^{-1}} = P_\mu.$$

By properties of symmetric operators, $\left\|P_\mu^k\right\| = \left\|P_\mu\right\|^k$ and so it suffices to show that $\left\|P_\mu\right\| \le \rho(\mu)$.

To show that $\left\|P_\mu\right\| \le \rho(\mu)$, it suffices to fix $f \in L^2(\Gamma)$ and show that $\left\|P_\mu f\right\|_2 \le \rho(\mu)\left\|f\right\|_2$. By density, we can assume that $f$ is finitely supported. Further, $\left\|P_\mu f\right\|_2 \le \left\|P_\mu |f|\right\|_2$ and so we can assume that $f \ge 0$.

So suppose $f \in L^2(\Gamma)$ is non-zero, has finite support, and $f \ge 0$. Notice that

$$\begin{aligned}\left\langle P_\mu^{2n} f, f\right\rangle &= \sum_x \sum_y f(xy) f(x) \mu^{*2n}(y) = \sum_{x \in \mathrm{supp}(f)} \sum_{y \in x^{-1}\cdot \mathrm{supp}(f)} f(xy) f(x) \mu^{*2n}(y) \\ &\le |\mathrm{supp}(\mu)|^2 \left\|f\right\|_\infty^2 \max_{y \in \mathrm{supp}(f)^{-1}\,\mathrm{supp}(f)} \mu^{*2n}(y).\end{aligned}$$

So Observation 26.9 implies that

$$\limsup_{n\to\infty} \left\|P_\mu^n f\right\|_2^{1/n} = \limsup_{n\to\infty} \left\langle P_\mu^{2n} f, f\right\rangle^{1/(2n)} \le \rho(\mu).$$

Further, since $P_\mu$ is symmetric,

$$\left\|P_\mu^n f\right\|_2^2 = \left\langle P_\mu^n f, P_\mu^n f\right\rangle = \left\langle P_\mu^{n-1} f, P_\mu^{n+1} f\right\rangle \le \left\|P_\mu^{n-1} f\right\|_2 \left\|P_\mu^{n+1} f\right\|_2.$$

So

$$\frac{\left\|P_\mu^{n+1} f\right\|_2}{\left\|P_\mu^n f\right\|_2}$$

is non-decreasing and hence has a limit $L$. Hence

$$\rho(\mu) \ge \limsup_{n\to\infty} \left(\frac{\left\|P_\mu^n f\right\|_2}{\left\|f\right\|_2}\right)^{1/n} = \limsup_{n\to\infty} \left(\prod_{j=0}^{n-1} \frac{\left\|P_\mu^{j+1} f\right\|_2}{\left\|P_\mu^j f\right\|_2}\right)^{1/n} = L.$$

Then

$$\frac{\left\|P_\mu f\right\|_2}{\left\|f\right\|_2} \le \lim_{n\to\infty} \frac{\left\|P_\mu^{n+1} f\right\|_2}{\left\|P_\mu^n f\right\|_2} = L \le \rho(\mu). \qquad \square$$

**Lemma 26.16.** *If $\Gamma$ is amenable and $\mu$ is symmetric, then $\rho(\mu) = 1$.*

*Proof.* By Theorem 26.8, there exists a sequence of unit vectors $\{f_n\} \subset L^2(\Gamma)$ such that $\|f_n - R_g f_n\|_2 \to 0$ for all $g \in \Gamma$. Further, $P_\mu = \sum_g \mu(g) R_g$ and so $\|f_n - P_\mu f_n\|_2 \to 0$. Since $\|f_n\|_2 = 1$, this implies that $\|P_\mu f\|_2 \to 1$ and so $\|P_\mu\| = 1$. □

In the proof of the converse direction we will need to turn estimates on $\|P_\mu f\|_2$ into estimates on $\|f - R_g f\|_2$. To do this we will exploit the strict convexity of the unit ball in $L^2(\Gamma)$.

**Lemma 26.17.** *For any $\delta > 0$ there exists $\epsilon > 0$ such that: If $f_1, f_2 \in L^2(\Gamma)$, $\|f_1\|_2 = \|f_2\|_2 = 1$, and $\|f_1 - f_2\|_2 \geq \delta$, then $\|f_1 + f_2\| \leq 2 - \epsilon$.*

*Proof.* Note

$$\delta^2 \leq \|f_1 - f_2\|_2^2 = 2 - 2\langle f_1, f_2 \rangle .$$

Hence $\langle f_1, f_2 \rangle \leq 1 - \frac{\delta^2}{2}$. Then

$$\|f_1 + f_2\|_2^2 = 2 + 2\langle f_1, f_2 \rangle \leq 4 - \delta^2.$$

So $\epsilon := 2 - \sqrt{4 - \delta^2}$ suffices.

□

**Lemma 26.18.** *If $\left\|P_\mu^k\right\| = 1$ for all $k \geq 1$ (e.g. $\rho(\mu) = 1$), then there exists a sequence of unit vectors $\{f_n\} \subset L^2(\Gamma)$ such that $\|f_n - R_g f_n\|_2 \to 0$ for all $g \in \Gamma$. Hence $\Gamma$ is amenable.*

*Proof.* Fix $k \geq 1$ such that $\mu^{*k}(\mathrm{id}) > 0$. By assumption $\left\|P_\mu^k\right\| = 1$ and hence there exists a sequence of unit vectors $\{h_n\} \subset L^2(\Gamma)$ such that $\left\|P_\mu^k h_n\right\|_2 \to 1$. Notice that

$$\left|P_\mu^k h\right| \leq P_\mu^k |h|$$

and so by replacing each $h_n$ with $|h_n|$, we can assume that $h_n \geq 0$ for all $n \geq 1$.

We claim that $\|R_{g_1} h_n - R_{g_2} h_n\|_2 \to 0$ for all $g_1, g_2 \in \mathrm{supp}(\mu^{*k})$. Suppose not. Then after passing to a subsequence, there exists $g_1, g_2 \in \mathrm{supp}(\mu^{*k})$ and $\delta > 0$ such that

$$\|R_{g_1} h_n - R_{g_2} h_n\|_2 \geq \delta$$

for all $n \geq 1$. We can assume that $\mu^{*k}(g_1) \leq \mu^{*k}(g_2)$. Let $\epsilon = \epsilon(\delta) > 0$ satisfy Lemma 26.17. Then

$$\begin{aligned}
\left\|P_\mu^k h_n\right\|_2 &= \left\| \sum_{g \in G} \mu^{*k}(g) R_g h_n \right\|_2 \\
&\leq \left\|\mu^{*k}(g_1) R_{g_1} h_n + \mu^{*k}(g_1) R_{g_0} h_n\right\|_2 + \left\|(\mu^{*k}(g_2) - \mu^{*k}(g_1)) R_{g_2} h_n\right\|_2 \\
&\qquad + \sum_{g \in \Gamma - \{g_1, g_2\}} \left\|\mu^{*k}(g) R_g h_n\right\|_2 \\
&\leq \mu^{*k}(g_1) \left\|R_{g_1} h_n + R_{g_0} h_n\right\|_2 + \mu^{*k}(g_2) - \mu^{*k}(g_1) + 1 - \mu^{*k}(g_2) - \mu^{*k}(g_1) \\
&\leq \mu^{*k}(g_1)(2 - \epsilon) + 1 - 2\mu^{*k}(g_1) = 1 - \epsilon \mu^{*k}(g_1).
\end{aligned}$$

Since $\left\|P_\mu^k h_n\right\|_2 \to 1$ we have a contradiction. Hence $\|R_{g_1} h_n - R_{g_2} h_n\|_2 \to 0$ for all $g_1, g_2 \in \mathrm{supp}(\mu^{*k})$.

Since $\mathrm{id} \in \mathrm{supp}(\mu^{*k})$, we have

$$\left\|P_\mu^k h_n - h_n\right\|_2 \leq \sum_{g\in\Gamma} \mu^{*k}(g) \left\|h_n - R_g h_n\right\|_2 \to 0.$$

by the dominated convergence theorem (note that $g \mapsto 2\mu^{*k}(g)$ is in $L^1(\Gamma)$).

Let

$$f_n := \frac{h_n + P_\mu h_n + \cdots + P_\mu^{k-1} h_n}{\left\|h_n + P_\mu h_n + \cdots + P_\mu^{k-1} h_n\right\|_2}$$

(since $h_n \geq 0$, the denominator is positive and so this is well-defined). Then $\|f_n - P_\mu f_n\|_2 \to 0$. Since $\|f_n\|_2 = 1$, we must have $\|P_\mu f_n\|_2 \to 1$. Then repeating the above argument, we have $\|R_{g_1} f_n - R_{g_2} f_n\|_2 \to 0$ for all $g_1, g_2 \in \mathrm{supp}(\mu)$. Thus if $g \in \mathrm{supp}(\mu)$, we have

$$0 = \lim_{n\to\infty} \|f_n - P_\mu f_n\|_2 = \lim_{n\to\infty} \|f_n - R_g f_n\|_2 .$$

Since $\mathrm{supp}(\mu)$ generates $\Gamma$, this completes the proof.

□

As a corollary to the proof we have the following.

**Corollary 26.19.** *If $\Gamma$ is non-amenable, then there exist $C > 1$ and $0 < \lambda < 1$ such that*

$$\mu^{*n}(g) \leq C\lambda^n$$

*for all $g \in \Gamma$.*

*Proof.* By Lemma 26.18, $\left\|P_\mu^k\right\| < 1$ for some $k \geq 1$. Then by Lemma 26.14,

$$\mu^{*n}(g) \leq \left\|P_\mu^n\right\| \leq \left\|P_\mu^k\right\|^{\lfloor n/k \rfloor} \|P_\mu\|^{n-k\lfloor n/k \rfloor} \leq \left\|P_\mu^k\right\|^{n/k-1} = C\lambda^n$$

where $C := \left\|P_\mu^k\right\|^{-1}$ and $\lambda := \left\|P_\mu^k\right\|^{1/k}$. □

## 27. Harmonic and superharmonic functions

Suppose $\Gamma$ is a finitely generated group and $\mu$ is a probability measure on $\Gamma$ so that the associated random walk is irreducible.

Given a function $f : \Gamma \to \mathbb{R}$, define $P_\mu f : \Gamma \to \mathbb{R}$ by

$$P_\mu f(x) = \sum_{g\in\Gamma} \mu(g) f(xg)$$

(assuming the sum exists for all $x$). Such a function is *$\mu$-harmonic* if $P_\mu f = f$ and *$\mu$-superharmonic* if $P_\mu f \leq f$. When the context is clear, we just write harmonic and superharmonic.

Our main goal for the rest of the class is to understand the set of bounded harmonic functions, which we denote by $\mathcal{H}^\infty(\Gamma, \mu)$.

### 27.1. Compactness of superharmonic functions.

We will frequently use the following estimate.

**Proposition 27.1** (Harnack's inequality)**.** *Suppose $\mathrm{d}_S$ is a word metric on $\Gamma$. There exists $\lambda > 1$ such that: if $f : \Gamma \to [0, \infty)$ is superharmonic and $g_1, g_2 \in \Gamma$, then*

$$\lambda^{-\mathrm{d}_S(g_1,g_2)} f(g_2) \leq f(g_1) \leq \lambda^{\mathrm{d}_S(g_1,g_2)} f(g_2).$$

*In particular, $f$ is either identically zero or positive everywhere.*

*Proof.* For every $s \in S$ fix $n(s) \in \mathbb{N}$ such that $\mu^{*n(s)}(s) > 0$. Let $\epsilon := \min_{s\in S} \mu^{*n(s)}(s)$ and let $\lambda := \epsilon^{-1}$.

If $f : \Gamma \to [0, \infty)$ superharmonic, $g \in \Gamma$, and $s \in S$, then

$$f(g) \geq P_\mu^{n(s)} f(g) = \sum_{x\in\Gamma} \mu^{*n(s)}(x) f(gx) \geq \epsilon f(gs)$$

and

$$f(gs) \geq P_\mu^{n(s^{-1})} f(gs) = \sum_{x\in\Gamma} \mu^{*n(s^{-1})}(x) f(gsx) \geq \epsilon f(g).$$

Now induct on $\mathrm{d}_S(g_1, g_2)$. □

Let

$$\mathcal{S}^+ := \{f : f \text{ is } \mu\text{-superharmonic, } f > 0, \text{ and } f(\mathrm{id}) = 1\}.$$

**Proposition 27.2.** *With the topology of pointwise convergence, $\mathcal{S}^+$ is compact.*

*Proof.* By Harnack's inequality, if $S$ is a finite symmetric generating set, then $f(g) \leq \lambda^{|g|_S}$ for all $f \in \mathcal{S}^+$. So we can view $\mathcal{S}^+$ as a subset of the compact space $\prod_{g\in\Gamma}[0, \lambda^{|g|_S}]$. Thus it suffices to show that $\mathcal{S}^+$ is closed. Suppose $f_n \to f$ pointwise and $\{f_n\} \subset \mathcal{S}^+$. Then $f \geq 0$ and $f(\mathrm{id}) = 1$. Further, by Fatou's

$$P_\mu f(x) \leq \liminf_{n\to\infty} P_\mu f_n(x) \leq \liminf_{n\to\infty} f_n(x) = f(x).$$

Hence $f$ is superharmonic and so by Harnack's inequality $f$ is positive everywhere. □

27.2. **The recurrent case.** In this section we show that for recurrent random walks, every bounded harmonic function is constant.

As always let $\Omega = \Gamma^{\mathbb{N}}$ and $\beta = \mu^{\otimes\mathbb{N}}$.

**Proposition 27.3.** *If $f : \Gamma \to \mathbb{R}$ is a bounded harmonic function, then*

$$\lim_{n\to\infty} f(g_1 \cdots g_n)$$

*exists for $\beta$-almost every $\mathbf{g} \in \Omega$.*

*Proof.* Let $\mathcal{F}_n$ be the sigma-algebra generated by cylinders of the form

$$A_1 \times \cdots \times A_n \times \Gamma^{\mathbb{N}}.$$

Then $\{\mathcal{F}_n\}$ is a filtration. Define $f_n : \Omega \to \mathbb{R}$ by $f_n(\mathbf{g}) = f(g_1 \cdots g_n)$. Since $f$ is bounded, $\sup_{n\geq 1} \|f_n\|_{L^1} < +\infty$. We claim that $\{f_n\}$ is a martingale. If $A = A_1 \times \cdots \times A_n \times \Gamma^{\mathbb{N}} \in \mathcal{F}_n$, then

$$\begin{aligned}
\int_A f_{n+m} d\beta &= \sum_{(g_1,\ldots,g_{n+m})\in A_1\times\cdots\times A_n\times\Gamma^m} f(g_1 \cdots g_{n+m})\mu(g_1)\cdots\mu(g_{n+m}) \\
&= \sum_{(g_1,\ldots,g_n)\in A_1\times\cdots\times A_n} P_\mu^m f(g_1 \cdots g_n)\mu(g_1)\cdots\mu(g_n) \\
&= \sum_{(g_1,\ldots,g_n)\in A_1\times\cdots\times A_n} f(g_1 \cdots g_n)\mu(g_1)\cdots\mu(g_n) = \int_A f_n d\beta.
\end{aligned}$$

Then by Doob's martingale convergence theorem (Theorem 9.4), $\lim_{n\to\infty} f(g_1 \cdots g_n)$ exists for almost every $\mathbf{g} \in \Omega$. □

Recall that $X_n$ denotes the $n^{th}$-step of the random walk starting at the identity.

**Lemma 27.4.** *If the random walk associated to $\mu$ is recurrent and $g \in \Gamma$, then*

$$\Pr(X_n = g \text{ infinitely often}) = 1.$$

*Proof.* Let $Z \subset \Omega$ be the set of random walks that never visit $g$. We claim that $Z$ has zero measure. Let $E_m$ be the random walks that return to the id for the first time at time $m$. Since the random walk is recurrent,

$$1 = \sum_m \beta(E_m).$$

Further, if $\pi_m : \Omega \to \Gamma^m$ is the projection onto the first $m$-coordinates, then $E_m = \pi_m(E_m) \times \Gamma^{\mathbb{N}}$ and hence

$$\beta(E_m) = \mu^{\otimes}(\pi_m(E_m)).$$

Since there is some path with positive probability that visits $g$ before returning to id, we have

$$\sum_m \mu^{\otimes m}(\pi_m(E_m \cap Z)) < 1.$$

Also, since almost every path returns to id, the subset

$$\sqcup_m \pi_m(E_m \cap Z) \times Z \subset Z$$

has full measure in $Z$. So

$$\beta(Z) = \beta(Z) \sum_m \mu^{\otimes m}(\pi_m(E_m \cap Z)),$$

which is only possible if $\beta(Z) = 0$.

Now let $\Omega' \subset \Omega$ be the set of random walks that return to the id infinitely often. Then

$$\begin{aligned} \Pr(X_n = g \text{ infinitely often}) &= \Pr(X_n = g \text{ for some } n \geq 1) \Pr(\Omega') \\ &= \Pr(X_n = g \text{ for some } n \geq 1) = 1. \end{aligned}$$

□

**Proposition 27.5.** *If the random walk associated to $\mu$ is recurrent, then every bounded harmonic function is constant.*

*Proof.* Suppose $f : \Gamma \to \mathbb{R}$ is bounded and harmonic. Fix $x, y \in \Gamma$. By the previous two lemmas,

$$\Pr\left(X_n = x \text{ infinitely often}, X_n = y \text{ infinitely often, and } \lim_{n\to\infty} f(X_n) \text{ exists}\right) = 1.$$

So

$$f(x) = \lim_{n\to\infty} f(X_n) = f(y) \quad \text{a.s.}$$

Thus $f$ is constant. □

## 28. Green's function and metric

Suppose $\Gamma$ is a finitely generated group and $\mu$ is a probability measure on $\Gamma$ so that the associated random walk is irreducible and transient. The *Green function* $G : \Gamma \times \Gamma \to \mathbb{R}$ is defined by

$$G(x, y) = \sum_{n=0}^{\infty} \mu^{*n}(x^{-1}y) = \sum_{n=0}^{\infty} p_n(x, y),$$

where $\mu^{*0} = \delta_{\mathrm{id}}$. Since the random walk is transient, $G$ is finite everywhere (see Proposition 25.2).

**Observation 28.1.** Let $G_y := G(\cdot, y) : \Gamma \to \mathbb{R}$. Then

$$(P_\mu G_y - G_y)(x) = \begin{cases} 0 & \text{if } x \neq y \\ -1 & \text{if } x = y \end{cases}.$$

Hence $x \mapsto G(x, y)$ is superharmonic.

*Proof.* Note

$$\begin{aligned} P_\mu G_y(x) &= \sum_{g\in\Gamma} \mu(g) G(xg, y) = \sum_{n=0}^{\infty} \sum_{g\in\Gamma} \mu(g)\mu^{*n}(g^{-1}x^{-1}y) = \sum_{n=0}^{\infty} \mu^{*(n+1)}(x^{-1}y) \\ &= G(x, y) - \mu^{*0}(x^{-1}y). \end{aligned}$$

□

Let

$$p(x, y) := \Pr(\text{a random walk starting at } x \text{ visits } y).$$

**Observation 28.2.**

$$p(x, y) := \frac{G(x, y)}{G(\mathrm{id}, \mathrm{id})}.$$

*Proof.* Since $p(x, y) = p(\mathrm{id}, x^{-1}y)$ and $G(x, y) = G(\mathrm{id}, x^{-1}y)$, we can assume $x = \mathrm{id}$. Further, $p(\mathrm{id}, \mathrm{id}) = 1$ and so we can assume $y \neq \mathrm{id}$.

As usual let $X_n$ denote the $n^{th}$-step of the random walk starting at the identity. Then define subsets

$$\begin{aligned} A_n &:= \{\mathbf{g} : X_j(\mathbf{g}) = y \text{ at least } n \text{ times}\}, \\ B_m &:= \{\mathbf{g} : X_m(\mathbf{g}) = y \text{ and } X_j(\mathbf{g}) \neq y \text{ when } n < m\}, \\ C_n &:= \{\mathbf{g} : X_j(\mathbf{g}) = \mathrm{id} \text{ at least } (n+1) \text{ times (including } j = 0)\} \end{aligned}$$

Then

$$A_n = \bigsqcup \pi_m(B_m) \times C_{n-1}$$

and so

$$\beta(A_n) = \beta(C_{n-1}) \sum_m \mu^{\otimes m}(\pi_m B_m).$$

Further,

$$p(\mathrm{id}, y) = \sum_m \beta(B_m) = \sum_m \mu^{\otimes m}(\pi_m B_m).$$

Finally, arguing as in Proposition 25.2,

$$\sum_{n=1}^{\infty} \beta(A_n) = \sum_{n=1}^{\infty} p_n(\mathrm{id}, y) = G(\mathrm{id}, y) - p_0(\mathrm{id}, y) = G(\mathrm{id}, y)$$

and

$$\sum_{n=0}^{\infty} \beta(C_n) = \sum_{n=0}^{\infty} p_n(\mathrm{id}, \mathrm{id}) = G(\mathrm{id}, \mathrm{id}).$$

□

The *Green "metric"* is defined by

$$\mathrm{d}_G(x, y) = -\log p(x, y) = -\log \frac{G(x, y)}{G(\mathrm{id}, \mathrm{id})}.$$

Since

$$p(x, y) \geq p(x, z)p(z, y),$$

we have

$$\mathrm{d}_G(x,y) \le \mathrm{d}_G(x,z) + \mathrm{d}_G(z,y)$$

for all $x, y, z \in \Gamma$.

**Example 28.3.** Suppose $\Gamma = (\mathbb{Z}, +)$ with $\mu = \frac{1}{3}\delta_{-1} + \frac{2}{3}\delta_1$. Then

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{3} \quad a.s.$$

So for any $n \ge 1$,

$$p(0,n) = 1.$$

Hence $\mathrm{d}_G(0,n) = 0$ for all $n \ge 1$.

**Lemma 28.4.**

*(1) If $\mathrm{d}_G(x,y) = 0$ and $\mathrm{d}_G(y,x) = 0$, then $x = y$. Hence, if the random walk is symmetric, then $\mathrm{d}_G$ is a metric on $\Gamma$.*

*(2) If $\Gamma$ is not isomorphic to $\mathbb{Z}$ and $\mathrm{d}_G(x,y) = 0$, then $x = y$.*

*Proof.* (1). Suppose $x \ne y$ and $\mathrm{d}_G(x,y) = \mathrm{d}_G(y,x) = 0$. Then

$$\begin{aligned}\Pr(\text{returns to id}) &\ge p(\mathrm{id}, x^{-1}y)p(x^{-1}y, \mathrm{id}) \\ &= p(x,y)p(y,x) = 1,\end{aligned}$$

which contradicts the fact that the random walk is transient. So (1) is true.

(2). We prove the contrapositive: If

$$Z := \{g : \mathrm{d}_G(\mathrm{id}, g) = 0, g \ne \mathrm{id}\} = \{g : p(\mathrm{id}, g) = 1, g \ne \mathrm{id}\}$$

is non-empty, then $\Gamma$ is isomorphic to $\mathbb{Z}$.

We first show that every element of $Z$ has infinite order. Suppose not. Fix $g \in Z$ with $g^n = \mathrm{id}$. Then

$$\Pr(X_n \text{ returns to id}) \ge p(\mathrm{id}, g)p(g, g^2)\cdots p(g^{n-1}, \mathrm{id}) = p(\mathrm{id}, g)^n = 1$$

which contradicts the fact that the random walk is transient. Thus the claim is true.

Fix $a \in Z$ with $|a|_{\mathrm{supp}(\mu)}$ minimal. Then

$$t := \Pr(X_n \text{ visits } a \text{ before any other element of } Z) > 0.$$

So if $b \in Z \smallsetminus \{a\}$, then

$$1 = p(\mathrm{id}, b) \le tp(a,b) + (1-t).$$

So $p(a,b) = 1$ and hence $a^{-1}b \in Z$.

Next we claim that $Z \subset a^{\mathbb{N}}$. Suppose not. Then fix $b \in Z$ with $b \ne a^n$ for all $n$. Then $a^{-n}b \in Z$ for all $n \ge 1$. On the other hand, since $a$ has infinite order, when $n$ is sufficiently large we have

$$\Pr(X_n \text{ visits } b \text{ before } a^n) > 0.$$

Then arguing as above, we must have $p(b, a^n) = 1$. Then

$$\Pr(X_n \text{ returns to id}) \ge p(\mathrm{id}, a^{-n}b)p(a^{-n}b, \mathrm{id}) = 1 \cdot p(b, a^n) = 1,$$

which contradicts the fact that the random walk is transient. Hence $Z \subset a^{\mathbb{N}}$.

Finally we claim that $\Gamma = a^{\mathbb{Z}}$. Since $\mathrm{supp}(\mu)$ generates $\Gamma$ it suffices to fix $s \in \mathrm{supp}(\mu)$ and show that $s \in a^{\mathbb{Z}}$. Since $s \in \mathrm{supp}(\mu)$,

$$\Pr(\text{visits } s \text{ before } a) > 0$$

and hence $p(s,a)=1$. Thus $s^{-1}a \in Z$. So $s^{-1}a \in a^{\mathbb{N}}$, which implies that $s \in a^{\mathbb{Z}}$. □

**Proposition 28.5.** *Suppose $\mathrm{d}_S$ is a word metric on $\Gamma$. If $\mu$ has finite support and $\Gamma$ is non-amenable, then there exist $A>1, B>0$ such that*

$$A^{-1}\,\mathrm{d}_S(x,y) - B \le \mathrm{d}_G(x,y) \le A\,\mathrm{d}_S(x,y) + B$$

*for all $x, y \in \Gamma$.*

*Proof.* Since $x \mapsto G(x,\mathrm{id})$ is superharmonic, by Harnack's inequality there exists $\lambda_1 > 1$ such that:

$$\lambda_1^{-|x|_S} G(\mathrm{id},\mathrm{id}) \le G(x,\mathrm{id})$$

for all $x \in \Gamma$. So

$$\begin{aligned}\mathrm{d}_G(x,y) &= -\log\frac{G(y^{-1}x,\mathrm{id})}{G(\mathrm{id},\mathrm{id})} \le (\log\lambda_1)\left|x^{-1}y\right|_S - \log G(\mathrm{id},\mathrm{id}) \\ &= (\log\lambda_1)\,\mathrm{d}_S(x,y) - \log G(\mathrm{id},\mathrm{id}).\end{aligned}$$

By Corollary 26.19, there exist $C_1 > 1$ and $0 < \lambda_2 < 1$ such that

$$\mu^{*n}(x) \le C_1\lambda_2^n$$

for all $n \ge 0$ and $x \in \Gamma$. Further, if $F := \mathrm{supp}(\mu)$, then

$$\mu^{*n}(x) = 0$$

when $n < |x|_F$. So

$$G(\mathrm{id},x) = \sum_{n=0}^{\infty}\mu^{*n}(x) \le C_1 \sum_{n=|x|_F}^{\infty}\lambda_2^n = \frac{C_1\lambda_2^{|x|_F}}{1-\lambda_2}.$$

Let $C_2 := \max_{g\in F}|g|_S$. Then $|x|_S \le C_2\,|x|_F$ and so

$$\begin{aligned}\mathrm{d}_G(x,y) &= -\log\frac{G(\mathrm{id},x^{-1}y)}{G(\mathrm{id},\mathrm{id})} \ge (-\log\lambda_2)\left|x^{-1}y\right|_F - B \\ &\ge \frac{-\log\lambda_2}{C_2}\,\mathrm{d}_S(x,y) - B\end{aligned}$$

where $B := \log\frac{C_1}{(1-\lambda_2)G(\mathrm{id},\mathrm{id})}$. □

## Part 5. The Martin Boundary

### 29. The Martin Boundary

Suppose $\Gamma$ is a finitely generated group and $\mu$ is a probability measure on $\Gamma$ so that the associated random walk is irreducible and transient. Recall that the set

$$\mathcal{S}^+ := \{f : f \text{ is } \mu\text{-superharmonic},\ f > 0,\ \text{and } f(\mathrm{id}) = 1\}$$

is compact with the topology of pointwise convergence.

Define $K(x,y) = \frac{G(x,y)}{G(\mathrm{id},y)}$. Observation 28.1 implies that

$$K(\cdot,y) \in \mathcal{S}^+$$

and hence the set $\{K(\cdot,y) : y \in \Gamma\}$ is relatively compact with respect to pointwise convergence. The *Martin boundary*, denoted $\partial_M\Gamma$, is the set of all functions $f : \Gamma \to \mathbb{R}$ where $f = \lim_{n\to\infty} K(\cdot,y_n)$ for some escaping sequence $\{y_n\}$ in $\Gamma$.

**Observation 29.1.** The set $\overline{\Gamma}^M := \Gamma \sqcup \partial_M \Gamma$ admits a topology which makes it a compactification of $\Gamma$, i.e. the inclusion $\Gamma \hookrightarrow \overline{\Gamma}^M$ is a topological embedding with dense image and $\overline{\Gamma}^M$ is a compact metrizable space. Further:

(1) A sequence $\{x_n\} \subset \Gamma$ converges to $f \in \partial_M \Gamma$ if and only if $\{x_n\}$ is escaping and $K(\cdot, x_n) \to f$ pointwise.
(2) $\{f_n\} \subset \partial_M \Gamma$ converges to $f \in \partial_M \Gamma$ if and only if $f_n \to f$ pointwise.
(3) The left action of $\Gamma$ on $\Gamma$ extends continuously to an action on $\overline{\Gamma}^M$ where $\Gamma$ acts on $\partial_M \Gamma$ by
$$g \cdot f = \frac{f \circ g^{-1}}{f(g^{-1})}.$$
(4) For any $x \in \Gamma$, the function $K(x, \cdot) : \Gamma \to \mathbb{R}_{>0}$ extends continuously to $K(x, \cdot) : \overline{\Gamma}^M \to \mathbb{R}_{>0}$ where
$$K(x, f) = f(x)$$
when $f \in \partial_M \Gamma$.

*Proof.* Enumerate $\Gamma = \{g_n\}$. By Harnack's inequality, there exists $C_n > 0$ such that
$$f(g_n) \leq C_n$$
for all $f \in \mathcal{S}^+$. Then define a distance $\rho_0$ on $\mathcal{S}^+$ by
$$\rho_0(f_1, f_2) = \sum_{n=1}^{\infty} \frac{1}{2^n C_n} |f_1(g_n) - f_2(g_n)|.$$
Then define a distance $\rho$ on $\overline{\Gamma}^M$ by
$$\begin{aligned} \rho(g_n, g_m) &= \max\{2^{-n}, 2^{-m}\}\delta_m(n) + \rho_0(K(\cdot, g_n), K(\cdot, g_m)) \\ \rho(g_n, f) &= \rho(f, g_n) = 2^{-n} + \rho_0(K(\cdot, g_n), f) \\ \rho(f_1, f_2) &= \rho_0(f_1, f_2). \end{aligned}$$
Check that this has all the desired properties. □

We will prove that any non-negative harmonic function can be realized as a convex combination of elements in the Martin boundary.

**Theorem 29.2.** *If $h : \Gamma \to [0, \infty)$ is $\mu$-harmonic, then there exists a measure $\nu_h$ on $\partial_M \Gamma$ such that*
$$h = \int_{\partial_M \Gamma} f d\nu_h(f). \tag{7}$$

To obtain a unique measure, we need to consider a subset of the Martin boundary. First a non-negative $\mu$-harmonic function $h$ is *minimal* if whenever $0 \leq u \leq h$ and $u$ is $\mu$-harmonic, then $u = Ch$ for some $C \in [0, 1]$. Then the *minimal Martin boundary* is
$$\partial_M^{\min} \Gamma := \{h \in \partial_M \Gamma : h \text{ is minimal } \mu\text{-harmonic}\}.$$
We will show that for every $\mu$-harmonic function $h$, there exists a unique measure $\nu_h$ on $\partial_M^{\min} \Gamma$ realizing Equation (7). Of particular importance is the measure $\nu := \nu_1$ corresponding to the constant 1 function. This measure has the following properties.

**Theorem 29.3.**

*(1)* $\nu$ *is a stationary measure supported on* $\partial_M^{\min}\Gamma$.

*(2) For* $\beta$*-almost every* $\mathbf{g}$,

$$\lim_{n\to\infty} g_1 \cdots g_n$$

*exists in* $\partial_M\Gamma$ *and if* $A \subset \partial_M\Gamma$ *is measurable, then*

$$\nu(A) = \Pr\left(\lim_{n\to\infty} g_1 \cdots g_n \in A\right).$$

*(3) Given* $\phi \in L^\infty(\partial_M\Gamma, \nu)$*, let*

$$H_\phi = \int f\phi(f) d\nu(f).$$

*The map* $\phi \mapsto H_\phi$ *is an isomorphism* $L^\infty(\partial_M\Gamma, \nu) \to \mathcal{H}^\infty(\Gamma, \mu)$.

## 30. Existence of a reproducing formula

Suppose $\Gamma$ is a finitely generated group and $\mu$ is a probability measure on $\Gamma$ so that the associated random walk is irreducible and transient.

In this section we prove Theorem 29.2.

**Theorem 30.1.** *If* $h : \Gamma \to [0, \infty)$ *is* $\mu$*-harmonic, then there exists a measure* $\nu_h$ *on* $\partial_M\Gamma$ *such that*

$$h = \int_{\partial_M\Gamma} f d\nu_h(f).$$

*Moreover for any such measure* $P_\mu f = f$ *for* $\nu_h$*-almost every* $f$.

30.1. **Potentials and the Riesz Representation Theorem.** A function $u : \Gamma \to [0, \infty)$ is a *potential* if

$$u = \sum_{y\in\Gamma} G(\cdot, y)k(y)$$

where $k : \Gamma \to [0, \infty)$. Since each $G(\cdot, y)$ is superharmonic, any potential is superharmonic.

**Theorem 30.2** (Riesz Representation Theorem)**.** *If* $u : \Gamma \to [0, \infty)$ *is superharmonic, then*

$$u = \sum_{y\in\Gamma} G(\cdot, y)k(y) + h$$

*where* $k, h \geq 0$ *and* $h$ *is harmonic. This decomposition is unique,* $k = u - P_\mu u$*, and* $h = \lim_{n\to\infty} P_\mu^n u$.

*Proof.* Since $P_\mu u \leq u$, we have $P_\mu^{n+1} u \leq P_\mu^n u$ for all $n$. Thus $h := \lim_{n\to\infty} P_\mu^n u$ exists. Further, by the monotone convergence theorem,

$$P_\mu h = \lim_{n\to\infty} P_\mu^{n+1} u = h.$$

So $h$ is harmonic. Then

$$u = P_\mu^m u + \sum_{n=0}^{m-1} P_\mu^n u - P_\mu^{n+1} u = P_\mu^n u(x) + \sum_{n=0}^{m-1} P_\mu^n (u - P_\mu u)$$

$$= P_\mu^m u + \sum_{y\in\Gamma} \sum_{n=0}^{m-1} p_n(\cdot, y)(u - P_\mu u)(y).$$

Then sending $m \to +\infty$, we obtain

$$u = h + \sum_{y \in \Gamma} G(\cdot, y)k(y)$$

where $k = u - P_\mu u$ and $h = \lim_{n\to\infty} P_\mu^n u$.

For uniqueness, suppose

$$u = \sum_{y \in \Gamma} G(\cdot, y)k_0(y) + h_0$$

where $k_0 \geq 0$ and $h_0$ is a non-negative harmonic function. Since $P_\mu G(\cdot, y) = G(\cdot, y) - \delta_y$, then

$$u(x) - k(x) = P_\mu u(x) = u(x) - k_0(x).$$

Hence $k = k_0$, which implies that $h = h_0$. □

**Corollary 30.3.** *If $u, v : \Gamma \to [0, \infty)$ are superharmonic, $v$ is a potential, and $u \leq v$, then $u$ is also a potential.*

*Proof.* Notice that

$$0 \leq \lim_{n\to\infty} P_\mu^n u \leq \lim_{n\to\infty} P_\mu^n v = 0.$$

□

30.2. **Proof of Theorem 30.1.** Suppose $h : \Gamma \to [0, \infty)$ is harmonic. Fix finite sets $S_1 \subset S_2 \subset \cdots \Gamma$ such that $\Gamma = \cup S_n$. Define

$$h_n(x) := \min \left\{ h(x), n \sum_{y \in S_n} G(x, y) \right\}$$

Then, since $G > 0$, we have $h_n \to h$ pointwise. Further, $n \sum_{y\in S_n} G(x, y) = \sum_{y\in\Gamma} G(x, y)n1_{S_n}(y)$ and hence is a potential. So by Corollary 30.3, each $h_n$ is a potential. So

$$h_n = \sum_{y \in \Gamma} G(x; y)k_n(y)$$

for some $k_n \geq 0$. Then

$$h_n(x) = \sum_{y \in \Gamma} K(x, y)G(\mathrm{id}, y)k_n(y) = \int_{\overline{\Gamma}^M} K(x, y)d\nu_n(y)$$

where $\nu_n := \sum_{y\in\Gamma} G(\mathrm{id}, y)k_n(y)\delta_y$. Notice that

$$\nu_n(\overline{\Gamma}^M) = \sum_{y \in \Gamma} G(\mathrm{id}, y)k_n(y) = h_n(\mathrm{id}) \leq h(\mathrm{id}).$$

Let $\nu_h$ be a weak-$*$ limit of $\{\nu_n\}$. Since $h_n \to h$ pointwise and $K(x, \cdot) : \overline{\Gamma}^M \to \mathbb{R}$ is continuous, we have

$$h(x) = \int_{\overline{\Gamma}^M} K(x, y)d\nu_h(y) = \sum_{y \in \Gamma} \frac{G(x, y)}{G(\mathrm{id}, y)} \nu_h(\{y\}) + \int_{\partial_M \Gamma} f(x)d\nu_h(f).$$

Since $h$ is harmonic and $G(\cdot, y) - P_\mu G(\cdot, y) = \delta_y$,

$$0 = h(x) - P_\mu h(x) = \frac{\nu_h(\{x\})}{G(\mathrm{id}, x)} + \int_{\partial_M \Gamma} f(x) - P_\mu f(x)d\nu_h(f) \geq \frac{\nu_h(\{x\})}{G(\mathrm{id}, x)}.$$

Thus $\nu_h$ has support in $\partial_M\Gamma$ and so

$$h = \int_{\partial_M\Gamma} f d\nu_h(f).$$

Now suppose that

$$h = \int_{\partial_M\Gamma} f d\lambda$$

for some measure $\lambda$. Fixing $x \in \Gamma$,

$$0 = (h - P_\mu h)(x) = \int_{\partial_M\Gamma} (f - P_\mu f)(x) d\lambda \leq 0$$

and hence $(f - P_\mu f)(x) = 0$ for $\lambda$-almost every $f$. Thus $f - P_\mu f \equiv 0$ for $\lambda$-almost every $f$.

## 31. Almost sure convergence of random walks

Suppose $\Gamma$ is a finitely generated group and $\mu$ is a probability measure on $\Gamma$ so that the associated random walk is irreducible and transient.

Let $\mathcal{W} = \Gamma^{\mathbb{N}}$ and for $x \in \Gamma$ consider the probability measures $\mathbb{P}_x$ on $\mathcal{W}$ which satisfies

$$\mathbb{P}_x((w_n) : w_1 = y_1, \dots, w_m = y_m) = \delta_x(y_1)\mu(y_1^{-1}y_2)\mu(y_2^{-1}y_3)\cdots\mu(y_{m-1}^{-1}y_m).$$

So

$$\mathbb{P}_x((w_n) : w_m = y)$$

represents the probability that a random walk starting at $x$ with step distribution $\mu$ is at $y$ at after $m$ steps. Also note that

$$\mathbb{P}_{\mathrm{id}} = \omega_* \mu^{\mathbb{N}}$$

where $\omega : \Omega \to \mathcal{W}$ is the map

$$\omega(\mathbf{g}) = (\mathrm{id},\, g_1,\, g_1g_2,\, g_1g_2g_3, \dots).$$

*Remark* 31.1. Notice that $\Omega$ and $\mathcal{W}$ coincide as sets, but we use different letters since $\Omega$ represents the steps of a random walk and $\mathcal{W}$ represents the locations of a random walk.

It will be helpful to consider more general random walks. Given a positive $\mu$-harmonic function $h : \Gamma \to (0, \infty)$, we define a family of probability measures $\{\mathbb{P}_x^h\}_{x\in\Gamma}$ on $\mathcal{W}$ by

$$\mathbb{P}_x^h(\mathbf{w} = (w_n) : w_1 = y_1, \dots, w_m = y_m) = \frac{h(y_m)}{h(x)}\delta_x(y_1)\mu(y_1^{-1}y_2)\mu(y_2^{-1}y_3)\cdots\mu(y_{m-1}^{-1}y_m).$$

This records the statistics for the random walk starting at $x$ where the step distribution at location $z$ is given by

$$\frac{h(y)}{h(z)}\mu(z^{-1}y).$$

Notice that, since $h$ is $\mu$-harmonic, we have

$$\sum_{y\in\Gamma} \frac{h(y)}{h(x)}\mu(x^{-1}y) = 1$$

for all $x \in \Gamma$. Hence each $\mathbb{P}_x^h$ is indeed a probability measure.

**Theorem 31.2.** *Suppose $h$ is a positive $\mu$-harmonic function. For $\mathbb{P}_x^h$-a.e. $(w_n) \in \mathcal{W}$ the limit*

$$\lim_{n\to\infty} w_n$$

*exists in $\overline{\Gamma}^M$ and is in $\partial_M^{\min}\Gamma$. Moreover, if $\nu_h$ is a measure on $\partial_M^{\min}\Gamma$ such that*

$$h = \int_{\partial_M^{\min}\Gamma} f d\nu_h(f),$$

*then*

$$\mathbb{P}_x^h\left(\lim_{n\to\infty} w_n \in A\right) = \frac{1}{h(x)}\int_A f(x) d\nu_h(f)$$

*for all measurable subsets $A \subset \partial_M^{\min}\Gamma$.*

The rest of the section is devoted to the proof of the theorem, which is adapted from [Saw97].

We first observe the following convexity result.

**Proposition 31.3.** *Suppose $h$ is a positive $\mu$-harmonic function and $\nu_h$ is a measure on $\partial_M\Gamma$ such that*

$$h = \int_{\partial_M\Gamma} f d\nu_h(f).$$

*Then for any measurable subset $B \subset \mathcal{W}$, we have*

$$\mathbb{P}_x^h(B)h(x) = \int_{\partial_M\Gamma} \mathbb{P}_x^f(B)f(x) d\nu_h(f).$$

*Remark* 31.4. By Theorem 30.1, $\nu_h$-a.e. $f$ is $\mu$-harmonic and so $\mathbb{P}_x^f$ is well-defined $\nu_1$-a.e.

*Proof.* Notice that the right side defines a Borel measure on $\mathcal{W}$ supported on $\{x\} \times \Gamma^{\mathbb{N}}$. Hence it suffices to consider sets of the form $B = \{(x, y_1, \dots, y_n)\} \times \Gamma^{\mathbb{N}}$. In this case,

$$\mathbb{P}_x^h(B)h(x) = \mu(x^{-1}y_1)\mu(y_1^{-1}y_2)\cdots\mu(y_{n-1}^{-1}y_n)h(y_n)$$

and

$$\begin{aligned}\int_{\partial_M\Gamma} \mathbb{P}_x^f(B)f(x)d\nu_1(f) &= \int_{\partial_M\Gamma} f(y_n)\mu(x^{-1}y_1)\mu(y_1^{-1}y_2)\cdots\mu(y_{n-1}^{-1}y_n)d\nu_1(f)\\ &= \mu(x^{-1}y_1)\mu(y_1^{-1}y_2)\cdots\mu(y_{n-1}^{-1}y_n)h(y_n)\end{aligned}$$

since $\int f d\nu_h(f) = h$.

□

The key lemma is the following.

**Lemma 31.5.** *If $h \in \partial_M^{\min}\Gamma$, then*

$$\lim_{n\to\infty} w_n = h$$

*for $\mathbb{P}_x^h$-a.e. $(w_n) \in \mathcal{W}$.*

Assuming the lemma for a moment, we prove Theorem 31.2.

*Proof of Theorem 31.2.* Let

$$B = \{(w_n) \in \mathcal{W} : \lim_{n\to\infty} w_n \text{ exists and is in } \partial_M^{\min}\Gamma\}.$$

Then by the Lemma and the Proposition,

$$\mathbb{P}_x^h(B)h(x) = \int_{\partial_M\Gamma} \mathbb{P}_x^f(B)f(x)d\nu_h(f) = \int_{\partial_M\Gamma} 1\cdot f(x)d\nu_h(f) = h(x).$$

This proves the first assertion. For the "moreover" part, fix a measurable subset $A \subset \partial_M^{\min}\Gamma$. Then

$$\begin{aligned}\mathbb{P}_x^h\left(\lim_{n\to\infty} w_n \in A\right) &= \frac{1}{h(x)}\int_{\partial_M\Gamma} \mathbb{P}_x^f\left(\lim_{n\to\infty} w_n \in A\right) f(x)d\nu_h(f) = \frac{1}{h(x)}\int_{\partial_M\Gamma} 1_A(f)\cdot f(x)d\nu_h(f)\\ &= \frac{1}{h(x)}\int_A f d\nu_h(f).\end{aligned}$$

□

31.1. **Proof of Lemma 31.5.** Fix $h \in \partial_M^{\min}\Gamma$. It suffices to fix $r > 0$ and a metric $\rho$ on $\overline{\Gamma}^M$ generating the topology, then show that

$$v(x) = \mathbb{P}_x^h\left((w_n) : \rho(w_n, h) > r \text{ for infinitely many } n\right)$$

is identically zero.

**Lemma 31.6.** *$v \equiv 0$ or $v \equiv 1$.*

*Proof.* The proof is somewhat complicated: we first show that $v\cdot h$ is $\mu$-harmonic, then use the minimality of $h$ to show that $v$ is constant, then show that $v$ is either 0 or 1.

Let $E := \{(w_n) : \rho(w_n, h) > r \text{ for infinitely many } n\}$. Then $E = \sqcup_{x\in\Gamma}\{x\}\times E$ and so

$$v(x) = \mathbb{P}_x^h(E) = \sum_{y\in\Gamma}\frac{h(y)}{h(x)}\mu(x^{-1}y)\,\mathbb{P}_y^h(E) = \sum_{y\in\Gamma}\frac{h(y)}{h(x)}\mu(x^{-1}y)v(y).$$

Thus $v\cdot h$ is $\mu$-harmonic.

By definition $0 \le v \le 1$. So $0 \le v\cdot h \le h$. Since $h$ is a minimal harmonic function, we then have that $v$ is constant. Let $v \equiv c$.

Fix $\epsilon > 0$ and $x \in \Gamma$. By the definition of the product sigma algebra, there exists a finite disjoint union $F = \sqcup C_i$ where $\mathbb{P}_x^h(F\Delta E) < \epsilon$ and each $C_i$ has the form

$$C_i = \{(y_{i,1}, \dots, y_{i,n_i})\}\times\Gamma^{\mathbb{N}}.$$

Then $C_i \cap E = \{(y_{i,1},\dots,y_{i,n_i})\}\times E$ and so

$$\begin{aligned}\mathbb{P}_x^h(F\cap E) &= \sum_i \mathbb{P}_x^h(C_i\cap E) = \sum_i \frac{h(y_{n_i})}{h(x)}\delta_x(y_1)\mu(y_1^{-1}y_2)\cdots\mu(y_{n_i-1}^{-1}y_{n_i})\,\mathbb{P}_{y_{n_i}}^h(E)\\ &= \sum_i \mathbb{P}_x^h(C_i)c = \mathbb{P}_x^h(F)c.\end{aligned}$$

Notice that

$$\mathbb{P}_x^h(F\cap E) \ge \mathbb{P}_x^h(E) - \mathbb{P}_x^h(F\Delta E) > c-\epsilon \quad\text{and}\quad \mathbb{P}_x^h(F) \le \mathbb{P}_x^h(E) + \mathbb{P}_x^h(F\Delta E) < c+\epsilon.$$

So

$$c - \epsilon < (c+\epsilon)c.$$

Since $\epsilon > 0$ was arbitrary, we then have $c \le c^2$, which implies that $c \in \{0, 1\}$. □

Now it suffices to assume that $v \equiv 1$ and obtain a contradiction.

**Lemma 31.7.** *If $F \subset \Gamma$ is finite and $\tau_F(\mathbf{w}) = \inf\{n : w_n \in F\}$, then there exists a function $k : F \to [0, \infty)$ such that*

$$h(x)\,\mathbb{P}_x^h(\tau_F < +\infty) = \sum_{y \in F} G(x, y)k(y)$$

*Proof.* Let $u(x) = h(x)\,\mathbb{P}_x^h(\tau_F < +\infty)$. Then

$$\begin{aligned} u(x) - P_\mu u(x) &= h(x)\,\mathbb{P}_x^h(\tau_F < +\infty) - \sum_{y \in \Gamma} \mu(x^{-1}y) f(y)\,\mathbb{P}_y^h(\tau_F < +\infty) \\ &= h(x)\left(\mathbb{P}_x^f(w_n \in F \text{ for some } n \geq 1) - \mathbb{P}_x^h(w_n \in F \text{ for some } n \geq 2)\right) \\ &= h(x)1_F(x)\,\mathbb{P}_x^f(w_n \notin F \text{ for all } n \geq 2) \geq 0. \end{aligned}$$

Hence $u$ is $\mu$-superharmonic and $u - P_\mu u$ has support in $F$. Using the Riesz Representation Theorem (Theorem 30.2) it suffices to show that $P_\mu^n u \to 0$ pointwise.

Notice that

$$\begin{aligned} P_\mu^n u(x) &= \sum_{y_1, \ldots, y_n \in \Gamma} \mu(x^{-1}y_1) \cdots \mu(y_{n-1}^{-1}y_n) h(y_n)\,\mathbb{P}_{y_n}^h(\tau_F < +\infty) \\ &= h(x)\,\mathbb{P}_x^h(w_k \in F \text{ for some } k \geq n+1). \end{aligned}$$

One can show that

$$h(x)\,\mathbb{P}_x^h(w_k \in F \text{ for some } k \geq n+1) \leq \left(\max_{y \in F} h(y)\right) \mathbb{P}_x(w_k \in F \text{ for some } k \geq n+1).$$

Hence by the transitivity of the original random walk, we have $P_\mu^n u \to 0$ pointwise. □

**Lemma 31.8.** *There exists a measure $\nu_0$ supported on $\{f \in \partial_M \Gamma : \rho(f, h) \geq r\}$ such that*

$$h = \int_{\partial_M \Gamma} f d\nu_0(f).$$

*Proof.* Fix finite sets $F_1 \subset F_2 \subset \cdots$ such that

$$\bigcup F_n = \{y \in \Gamma : \rho(y, h) > r\} =: F.$$

By the previous lemma, for each $n$, there exists a function $k_n : F_n \to [0, \infty)$ such that

$$h(x)\,\mathbb{P}_x^h(\tau_{F_n} < +\infty) = \sum_{y \in F_n} G(x, y)k_n(y).$$

Consider the measure $\nu_n := \sum_{y \in F_n} G(\mathrm{id}, y)k_n(y)\delta_y$ on $\overline{\Gamma}^M$. Notice that

$$\nu_n(\overline{\Gamma}^M) = \sum_{y \in F_n} G(\mathrm{id}, y)k_n(y) = \sum_{y \in F_n} G(\mathrm{id}, y)k_n(y) = h(\mathrm{id})\,\mathbb{P}_{\mathrm{id}}^h(\tau_{F_n} < +\infty) \leq 1.$$

So we can pass to a subsequence and suppose that $\nu_n \to \nu_0$ in the weak-$*$ topology. Since the original random walk is transitive, $\nu_0$ is supported on $\partial_M \Gamma$. Then, by construction, $\nu_0$ is supported on $\{f \in \partial_M \Gamma : \rho(f, h) \geq r\}$.

Notice that $\tau_{F_n} \searrow \tau_F$. Further, since $v \equiv 1$, $\mathbb{P}_x^h(\tau_F < +\infty) = 1$. So

$$1 = \mathbb{P}_x^h(\tau_F < +\infty) = \lim_{n \to \infty} \mathbb{P}_x^h(\tau_{F_n} < +\infty).$$

Then by the continuity of $K(x,\cdot):\overline{\Gamma}^M \to [0,\infty)$, we have

$$h(x) = \lim_{n\to\infty} h(x)\,\mathbb{P}_x^h(\tau_{F_n} < +\infty) = \int_{\overline{\Gamma}^M} K(x,\cdot)d\nu_n = \int_{\partial_M\Gamma} f(x)d\nu_0(f). \qquad \square$$

Notice that

$$1 = h(\mathrm{id}) = \int_{\partial_M\Gamma} f(\mathrm{id})d\nu_0(f) = \nu_0(\partial_M\Gamma).$$

So $\nu_0$ is a probability measure. Then, since $f \in \partial_M^{\min}\Gamma$, we must have $\nu_0 = \delta_f$. However, this is impossible since $\nu_0$ is supported on $\{f \in \partial_M\Gamma : \rho(f,h) \geq r\}$. So we have a contradiction.

## 32. Uniqueness of the reproducing formula

Suppose $\Gamma$ is a finitely generated group and $\mu$ is a probability measure on $\Gamma$ so that the associated random walk is irreducible and transient.

Our goal this section is to prove the following theorem.

**Theorem 32.1.** *If $h : \Gamma \to [0,\infty)$ is $\mu$-harmonic, then there exists a unique measure $\nu_h$ supported on $\partial_M^{\min}\Gamma$ such that*

$$h = \int_{\partial_M\Gamma} f d\nu_h(f). \tag{8}$$

Recall that the set

$$\mathcal{S}^+ := \{f : f \text{ is } \mu\text{-superharmonic, } f > 0, \text{ and } f(\mathrm{id}) = 1\}$$

is compact with the topology of pointwise convergence. In fact $\mathcal{S}^+$ is a compact convex set in a reasonable vector space (the vector space of sequence $\mathbb{R}^\Gamma$ with the product topology). So if $\mathcal{E}$ is the set of extreme points of $\mathcal{S}^+$, then by Choquet's theorem for any $u \in \mathcal{S}^+$ there exists a probability measure $\nu_u$ with $\nu_u(\mathcal{E}) = 1$ such that

$$u = \int f d\nu_u(f).$$

We start by relating $\mathcal{E}$ and $\partial_M^{\min}\Gamma$.

**Lemma 32.2.** *Suppose $h : \Gamma \to [0,\infty)$ is a $\mu$-harmonic function with $h(\mathrm{id}) = 1$. Then $h \in \mathcal{E}$ if and only if $h \in \partial_M^{\min}\Gamma$.*

*Proof.* Assume $h \in \mathcal{E}$. Suppose $0 \leq u \leq h$ where $u$ is harmonic. If $u \equiv 0$ or $u \equiv h$, then $u = Ch$ for some $C \in [0,1]$. So we can assume that $u$ and $h-u$ are not identically zero. Then $u(\mathrm{id})$ and $(h-u)(\mathrm{id})$ are non-zero. So $\frac{u}{u(\mathrm{id})}, \frac{h-u}{(h-u)(\mathrm{id})} \in \mathcal{S}^+$. Further,

$$h = u(\mathrm{id})\frac{u}{u(\mathrm{id})} + (1-u(\mathrm{id}))\frac{h-u}{(h-u)(\mathrm{id})}.$$

Since $h \in \mathcal{E}$, then $h = \frac{u}{u(\mathrm{id})} = \frac{h-u}{(h-u)(\mathrm{id})}$. Thus $u = u(\mathrm{id})h$. So $h$ is minimal. By Theorem 30.1, $h = \int_{\partial_M\Gamma} f d\nu_h$ for some measure $\nu_h$ on $\partial_M\Gamma$. Since

$$1 = h(\mathrm{id}) = \int_{\partial_M\Gamma} f(\mathrm{id})d\nu_h = \int_{\partial_M\Gamma} d\nu_h = \nu_h(\partial_M\Gamma)$$

we see that $\nu_h$ is a probability measure. Since $\partial_M\Gamma \subset \mathcal{S}^+$ and $h \in \mathcal{E}$, we must have $\nu_h = \delta_h$. Hence $h \in \partial_M\Gamma$ and so $h \in \partial_M^{\min}\Gamma$.

Assume $h \in \partial_M^{\min}\Gamma$. Suppose $h = tu_1 + (1-t)u_2$ for some $u_1, u_2 \in \mathcal{S}^+$. Then

$$0 = (h - P_\mu h) = t(u_1 - P_\mu u_1) + (1-t)(u_2 - P_\mu u_2) \geq 0$$

which implies that $u_1, u_2$ are harmonic. Then by minimality and the fact that $h(\mathrm{id}) = u_1(\mathrm{id}) = u_2(\mathrm{id}) = 1$, we see that $h = u_1 = u_2$. Thus $h \in \mathcal{E}$.

□

*Proof of Theorem 32.1.* Suppose $h : \Gamma \to [0, \infty)$ is harmonic. If $h \equiv 0$, then the zero measure is the unique measure satisfying Equation (8). So suppose $h$ is not identically zero. Then by scaling we can assume that $h(\mathrm{id}) = 1$. Then $h \in \mathcal{S}^+$ and so there exists a probability measure $\nu_h$ supported on $\mathcal{E}$ such that

$$h = \int_{\mathcal{E}} f d\nu_h(f).$$

Notice that

$$0 = (h - P_\mu h)(x) = \int_{\mathcal{E}} (f - P_\mu f)(x) d\nu_h(f) \geq 0$$

and so $\nu_h$-almost every $f$ is harmonic. Thus, by the lemma, $\nu_h$ is supported on $\partial_M^{\min}\Gamma$.

Now by Theorem 31.2, for any measurable subset $A \subset \partial_M^{\min}\Gamma$ we have

$$\nu_h(A) = \frac{1}{h(\mathrm{id})} \int_A f(\mathrm{id}) d\nu_h(f) = \mathbb{P}_{\mathrm{id}}^h \left( \lim_{n\to\infty} w_n \in A \right).$$

Hence $\nu_h$ is unique. □

## 33. A stationary measures on the Martin boundary

Suppose $\Gamma$ is a finitely generated group and $\mu$ is a probability measure on $\Gamma$ so that the associated random walk is irreducible and transient. In this section we prove Theorem 29.3.

Since $1_\Gamma$ is $\mu$-harmonic, by Theorem 32.1 there exists a unique probability measure $\nu$ supported on $\partial_M^{\min}\Gamma$ such that

$$1_\Gamma = \int_{\partial_M \Gamma} f d\nu.$$

Theorem 31.2 applied to the function $h \equiv 1$ immediately implies part (2) of Theorem 29.3. We prove the other two parts in the next two lemmas.

**Lemma 33.1** (Theorem 29.3 part (1))**.** *For every $g \in \Gamma$, the measures $g_*\nu$, $\nu$ are absolutely continuous and*

$$\frac{dg_*\nu}{d\nu}(f) = f(g).$$

*Moreover, $\nu$ is a stationary measure.*

*Proof.* Since $\Gamma$ maps $\partial_M^{\min}\Gamma$ to $\partial_M^{\min}\Gamma$, we have $\mathrm{supp}(g_*\nu) \subset \partial_M^{\min}\Gamma$. Further,

$$\int f \frac{dg_*\nu}{f(g)} = \int g \cdot f \frac{d\nu}{(g \cdot f)(g)} = \int \frac{f \circ g^{-1}}{f(g^{-1})} \frac{f(g^{-1})}{f(g^{-1}g)} d\nu = \int f \circ g^{-1} d\nu = 1_\Gamma \circ g^{-1} = 1_\Gamma.$$

So by uniqueness, $d\nu(f) = \frac{1}{f(g)} dg_*\nu(f)$.

Since $\nu$-almost every $f$ is harmonic,

$$\frac{d(\mu * \nu)}{d\nu}(f) = \sum_{g\in\Gamma} \mu(g) \frac{dg_*\nu}{d\nu}(f) = \sum_{g\in\Gamma} \mu(g) f(g) = f(\mathrm{id}) = 1$$

almost everywhere. Hence $\mu * \nu = \nu$ and so $\nu$ is stationary. □

**Lemma 33.2** (Theorem 29.3 part (3))**.** *Given $\phi \in L^\infty(\partial_M\Gamma, \nu)$, define $H_\phi : \Gamma \to \mathbb{R}$ by*

$$H_\phi(g) = \int f(g)\phi(f)d\nu(f).$$

*The map $\phi \mapsto H_\phi$ is an isomorphism $L^\infty(\partial_M\Gamma, \nu) \to \mathcal{H}^\infty(\Gamma, \mu)$.*

*Proof.* We first check that $\phi \mapsto H_\phi$ maps $L^\infty(\partial_M\Gamma, \nu)$ into $\mathcal{H}^\infty(\Gamma, \mu)$. Fix $\phi \in L^\infty(\partial_M\Gamma, \nu)$. Since

$$H_\phi(g) = \int f(g)\phi(f)d\nu(f)$$

and $\nu$-almost-every $f$ is harmonic, using Fubini we see that $H_\phi$ is harmonic. Also, since $1_\Gamma = \int f d\nu$, we have

$$-\|\phi\|_\infty = -\|\phi\|_\infty \int f d\nu \leq H_\phi \leq \|\phi\|_\infty \int f d\nu = \|\phi\|_\infty .$$

Thus $H_\phi \in \mathcal{H}^\infty(\Gamma, \mu)$.

Onto: Fix $h \in \mathcal{H}^\infty(\Gamma, \mu)$. By replacing $h$ with $ah + b$ for some $a, b \in \mathbb{R}$, we can suppose that $0 < h < 1$. By Theorem 32.1, there exists a unique measure $\nu_h$ supported on $\partial_M^{\min}\Gamma$ such that

$$h = \int f d\nu_h.$$

Likewise, there exists a unique measure $\nu_{1-h}$ supported on $\partial_M^{\min}\Gamma$ such that

$$1 - h = \int f d\nu_{1-h}.$$

Now, by uniqueness, $\nu = \nu_h + \nu_{1-h}$. Hence $\nu_h, \nu_{1-h} \ll \nu$ and so $d\nu_h = \phi_1 d\nu$, $d\nu_{1-h} = \phi_2 d\nu$ for some $\phi_1, \phi_2 \geq 0$. Then $\phi_1 + \phi_2 = 1$ and so $0 \leq \phi_1 \leq 1$. Then $h = H_{\phi_1}$ and $\phi_1 \in L^\infty(\partial_M\Gamma, \nu)$.

One-to-One: By the uniqueness in Theorem 32.1, $\phi \mapsto H_\phi$ is one-to-one on the subset of non-negative functions. Suppose $H_{\phi_1} = H_{\phi_2}$. Let $c = \max\{\|\phi_1\|_\infty , \|\phi_2\|_\infty\}$. Then

$$H_{\phi_1+c} = H_{\phi_1} + c1_\Gamma = H_{\phi_2} + c1_\Gamma = H_{\phi_2+c}.$$

Since $\phi_1 + c, \phi_2 + c$ are both non-negative, then $\phi_1 + c = \phi_2 + c$ and so $\phi_1 = \phi_2$. □

# Part 6. The Martin Boundary of a word hyperbolic group

## 34. Ancona's estimate

Suppose $\Gamma$ is a non-elementary word hyperbolic group, $\mathrm{d}_S$ is a word metric on $\Gamma$, and $X$ is the Cayley graph associated to $X$ and $S$. The *Gromov compactification* of $\Gamma$ is

$$\overline{\Gamma}^{\mathcal{G}} := \Gamma \sqcup \partial_{\mathcal{G}} X.$$

This compactification of $\Gamma$ is independent of the choice of $S$, see [BH99, Chapter III.H].

Next suppose that $\mu$ is a finitely supported probability measure on $\Gamma$ so that the associated random walk is irreducible. Our goal over the next several sections is to prove that the Martin compactification associated to $\mu$ is homeomorphic to the Gromov compactification. This result is originally due to Ancona [Anc88, Anc87],

but the proof we provide is based on work of Gekhtman–Gerasimov–Potyagailo–Yang [GGPY21].

We start by observing that it suffices to show that the Green's function is coarsely sub-multiplicative along geodesics in $X$.

**Proposition 34.1** (Ancona)**.** *Assume the Green function has the following property: for every $R > 0$ there exists $C = C(R) > 1$ such that*

$$G(x, z) \leq CG(x, y)G(y, z)$$

*whenever $x, y, z \in \Gamma$ and $y$ is in the $R$-neighborhood of a geodesic segment joining $x$ to $z$.*

*Then the identity map $\Gamma \to \Gamma$ extends to a homeomorphism*

$$\overline{\Gamma}^M \to \overline{\Gamma}^{\mathcal{G}}.$$

*Remark* 34.2. Recall that

$$\frac{G(x, y)}{G(\mathrm{id}, \mathrm{id})}$$

represents the probability that a random walk starts at $x$ and visits $y$. Hence we always have

$$G(x, y)G(y, z) \leq G(x, z)G(\mathrm{id}, \mathrm{id}). \tag{9}$$

The rest of the section is devoted to the proof of the proposition.

**Lemma 34.3.** *Every element of $\partial_M \Gamma$ is harmonic.*

*Proof.* Fix $f \in \partial_M \Gamma$ and an escaping sequence $\{y_n\} \subset \Gamma$ with $K(\cdot, y_n) \to f$. Since $\mu$ has finite support, $P_\mu K(\cdot, y_n) \to P_\mu f$. Then $f$ is harmonic, since

$$P_\mu K(\cdot, y) = K(\cdot, y) + \frac{1}{G(\mathrm{id}, y)}\delta_y. \qquad \square$$

**Lemma 34.4.** *If $K_1, K_2 \subset \overline{\Gamma}^{\mathcal{G}}$ are disjoint compact sets, then there exists $C = C(K_1, K_2) > 0$ such that*

$$K(x, y) \leq CG(x, \mathrm{id})$$

*for all $x \in K_1 \cap \Gamma$ and $y \in K_2 \cap \Gamma$.*

*Proof.* By the visibility lemma (Proposition 20.14) there exists $R > 0$ such that any geodesic joining a point in $K_1 \cap \Gamma$ to a point in $K_2 \cap \Gamma$ intersects $B_R(\mathrm{id})$. Let $C = C(R) > 0$ satisfy Proposition 34.1. Then if $x \in K_1 \cap \Gamma$ and $y \in K_2 \cap \Gamma$, then id is the $R$-neighborhood of a geodesic joining $x$ to $y$. Hence

$$K(x, y) = \frac{G(x, y)}{G(\mathrm{id}, y)} \leq CG(x, \mathrm{id}).$$

$\square$

Recall, that $-\log G(x, y)$ is quasi-isometric to the word metric (Proposition 28.5) and so

$$\lim_{x \to \infty} G(x, \mathrm{id}) = 0.$$

**Lemma 34.5.** *If $\{x_n\} \subset \Gamma$ converges to $f \in \partial_M \Gamma$ and $\{x_n\}$ converges to $\xi \in \partial_{\mathcal{G}} \Gamma$, then*

$$\lim_{z \to \eta} f(z) = 0$$

*for all $\eta \in \partial_{\mathcal{G}} \Gamma$ distinct from $\xi$.*

*Proof.* By the previous lemma, there exists $C = C(\eta) > 0$ such that

$$\lim_{z\to\eta} f(z) = \lim_{z\to\eta}\lim_{n\to\infty} K(z, x_n) \leq C \lim_{z\to\eta} G(z, \mathrm{id}) = 0. \qquad \square$$

**Lemma 34.6.** *If $\{x_n\} \subset \Gamma$ converges in $\overline{\Gamma}^M$, then $\{x_n\}$ converges in $\overline{\Gamma}^{\mathcal{G}}$. Hence, the identity map $\Gamma \to \Gamma$ extends to a continuous map*

$$F : \overline{\Gamma}^M \to \overline{\Gamma}^{\mathcal{G}}.$$

*Proof.* Suppose the first assertion fails. Then there exist $\{x_n\}, \{y_n\} \subset \Gamma$ such that $x_n, y_n \to f \in \partial_M\Gamma$, $x_n \to \xi_1 \in \partial_G\Gamma$, $y_n \to \xi_2 \in \partial_{\mathcal{G}}\Gamma$, and $\xi_1 \neq \xi_2$. If $\eta \in \partial_{\mathcal{G}}\Gamma$, then either $\eta \neq \xi_1$ or $\eta \neq \xi_2$. So by the previous lemma

$$\lim_{z\to\eta} f(z) = 0.$$

Since $f > 0$ and

$$\lim_{z\to\eta} f(z) = 0$$

for all $\eta \in \partial_{\mathcal{G}}\Gamma$, $f$ must obtain its maximum value. Then since $P_\mu f = f$ and the random walk is irreducible, $f$ must be constant (check!). Since $\lim_{z\to\eta} f(z) = 0$ for any $\eta \in \partial_{\mathcal{G}}\Gamma$, we must have $f \equiv 0$. However, by definition $f(\mathrm{id}) = 1$ and so we have a contradiction. Thus the first assertion is true.

Define $F : \overline{\Gamma}^M \to \overline{\Gamma}^{\mathcal{G}}$ by

$$F(x) = \begin{cases} x & \text{if } x \in \Gamma \\ \lim\limits_{\Gamma \ni y \to x} y & \text{if } x \in \partial_M\Gamma \end{cases}.$$

The first assertion implies that $F$ is continuous (check!).

$\square$

**Lemma 34.7.** *If $\xi \in \partial_{\mathcal{G}}\Gamma$, then*

$$F^{-1}(\xi) = \overline{\mathrm{ConvHull}(F^{-1}(\xi) \cap \partial_M^{\min}\Gamma)}.$$

*Proof.* Fix $h \in F^{-1}(\xi)$. Then

$$h = \int f d\nu_h$$

for some probability measure $\nu_h$ supported on $\partial_M^{\min}\Gamma$. It suffices to show that $\nu_h(F^{-1}(\xi)) = 1$.

Then it suffices to fix a closed set $A \subset \partial_M\Gamma$ where $A \cap F^{-1}(\xi) = \emptyset$ and show that $\nu_h(A) = 0$. Let

$$h' := \int_A f d\nu_h.$$

Then $h' \leq h$. By Lemma 34.4, there exists $C > 0$ such that

$$\lim_{x\to\xi} h'(x) = \lim_{x\to\xi} \int_A f(x) d\nu_h \leq \lim_{x\to\xi} CG(x, \mathrm{id}) = 0.$$

However, if $\eta \neq \xi$, then by Lemma 34.5 applied to $h$,

$$\lim_{y\to\eta} h'(y) \leq \lim_{y\to\eta} h(y) = 0.$$

Thus $h'$ realizes its maximum and hence is constant. Thus $h' \equiv 0$ and so $\nu_h(A) = h'(\mathrm{id}) = 0$.

$\square$

**Lemma 34.8.** *$F$ is injective and hence a homeomorphism.*

*Proof.* It suffices to fix $\xi \in \partial_{\mathcal{G}}\Gamma$ and show that $F^{-1}(\xi)$ is a single point. Recall that $X$ is a Cayley graph of $\Gamma$. Fix a geodesic ray $\ell : [0,\infty) \to X$ with $\ell(0) = \mathrm{id}$ and $[\ell] = \xi$. Then fix $\{m_j\} \subset \mathbb{N}$ such that $m_j \to \infty$ and $K(\cdot, \ell(m_j)) \to h_0 \in \partial_M\Gamma$.

**Claim:** $F^{-1}(\xi) \cap \partial_M^{\min}\Gamma = \{h_0\}$ and hence

$$F^{-1}(\xi) = \overline{\mathrm{ConvHull}(F^{-1}(\xi) \cap \partial_M^{\min}\Gamma)} = \{h_0\}.$$

By the previous lemma there exists some $h \in F^{-1}(\xi) \cap \partial_M^{\min}\Gamma$ . Fix a sequence $\{y_n\}$ converging to $h$ in $\overline{\Gamma}^M$. Then $y_n$ converges to $\xi = F(h)$ in $\overline{\Gamma}^{\mathcal{G}}$. Let $\ell_n$ be a geodesic joining id to $y_n$. Then passing to a subsequence we can suppose that $\ell_n \to \bar{\ell}$. Then $[\bar{\ell}] = \xi = [\ell]$ and so

$$R := \sup_{t \geq 0} \mathrm{d}(\ell(t), \bar{\ell}(t)) < +\infty.$$

Fix $j$ and fix $n$ sufficiently large so that

$$\mathrm{d}(\ell(m_j), \ell_n(m_j)) < R + 1.$$

Then $\ell(m_j)$ is in the $(R+1)$-neighborhood of $\ell_n$ and hence

$$G(\mathrm{id}, y_n) \leq CG(\mathrm{id}, \ell(m_j))G(\ell(m_j), y_n)$$

for some $C > 1$. By Equation (9)

$$G(x, y_n) \geq G(\mathrm{id}, \mathrm{id})^{-1} G(x, \ell(m_j))G(\ell(m_j), y_n)$$

and so

$$K(x, y_n) = \frac{G(x, y_n)}{G(\mathrm{id}, y_n)} \geq \frac{G(\mathrm{id}, \mathrm{id})^{-1} G(x, \ell(m_j))G(\ell(m_j), y_n)}{CG(\mathrm{id}, \ell(m_j))G(\ell(m_j), y_n)} = \frac{G(\mathrm{id}, \mathrm{id})^{-1}}{C} K(x, \ell(m_j)).$$

Then sending $n \to \infty$,

$$h(x) \geq \frac{1}{CG(\mathrm{id}, \mathrm{id})} K(x, \ell(m_j)).$$

Then sending $j \to \infty$,

$$h(x) \geq \frac{1}{CG(\mathrm{id}, \mathrm{id})} h_0(x)$$

for all $x \in \Gamma$.

Since $h$ is a minimal harmonic function and $h(\mathrm{id}) = 1 = h_0(\mathrm{id})$, we have $h = h_0$. Then $F^{-1}(\xi) \cap \partial_M^{\min}\Gamma = \{h_0\}$. Thus the claim is true.

□

## 35. Visual metrics

Suppose $X$ is Gromov hyperbolic. For $x, y, o \in X$, the *Gromov product of $x, y$ based at $o$* is

$$(x|y)_o := \frac{1}{2} \left( \mathrm{d}(x, o) + \mathrm{d}(o, y) - \mathrm{d}(x, y) \right).$$

Using the Gromov product, we define a metic on $\overline{X}^{\mathcal{G}}$. First we extend the Gromov product to $\overline{X}^{\mathcal{G}} \times \overline{X}^{\mathcal{G}}$ by taking limits. For $\xi, \eta \in \partial_{\mathcal{G}} X$, define

$$(\xi|\eta)_o := \sup \left\{ \limsup_{n \to \infty} (x_n|y_n)_o \right\}$$

where the supremum is taken over all sequence $(x_n), (y_n) \subset X$ with $x_n \to \xi$ and $y_n \to \eta$. Next, for $\xi \in \partial_{\mathcal{G}} X$ and $x \in X$, define

$$(\xi|x)_o = (x|\xi)_o := \sup \left\{ \limsup_{n\to\infty} (y_n|x)_o \right\}$$

where the supremum is taken over all sequence $(y_n) \subset X$ with $y_n \to \xi$.

Next fix $\epsilon > 0$ sufficiently small and define $\mathrm{D}_v : \overline{X}^{\mathcal{G}} \times \overline{X}^{\mathcal{G}} \to [0, \infty)$ by

$$\mathrm{D}_v(x, y) = \inf \left\{ \sum_{i=0}^{N-1} e^{-\epsilon\,(x_i|x_{i+1})_o} \right\}$$

where the infimum is taken over all finite lists $x_0, \ldots, x_N$ in $\overline{X}^{\mathcal{G}}$ with $x_0 = x$ and $x_N = y$. Finally, define $\overline{\mathrm{D}}_v : \overline{X}^{\mathcal{G}} \times \overline{X}^{\mathcal{G}} \to [0, \infty)$ by

$$\overline{\mathrm{D}}_v(x, y) := \min \{\epsilon\, \mathrm{d}(x, y), \mathrm{D}_v(x, y)\}$$

where $\mathrm{d}(x, y) = +\infty$ when $x \in \partial_{\mathcal{G}} X$ or $x \in \partial_{\mathcal{G}} X$.

**Theorem 35.1.** *For any $o \in X$ and $\epsilon > 0$ sufficiently small the function $\overline{\mathrm{D}}_v$ is a metric on $\overline{X}^{\mathcal{G}}$ which generates the standard topology. Moreover, there exists $C > 1$ such that: If $x, y \in X$, then*

$$C^{-1} e^{-\epsilon(x|y)_o} \le \mathrm{D}_v(x, y) \le C e^{-\epsilon(x|y)_o}.$$

The proof is somewhat involved and for details one can look at [DSU17, Section 3.6.2].

35.1. **Geometric meaning of the Gromov product.** The Gromov product has the following geometric meaning.

**Proposition 35.2.** *Suppose $X$ is $\delta$-hyperbolic and $x, y, o \in X$.*

*(1) If $\ell$ is a geodesic segment joining $x$ to $y$, then*

$$\mathrm{d}(o, \ell) - 2\delta < (x|y)_o \le \mathrm{d}(o, \ell).$$

*(2) If $\ell_x, \ell_y$ are geodesic segments joining $o$ to $x, y$ respectively and*

$$T := \max\{t \ge 0 : \mathrm{d}(\ell_x(t), \ell_y(t)) \le 2\delta\},$$

*then*

$$T - \delta \le (x|y)_o \le T + \delta.$$

*Proof.* (1). If $u \in \ell$, then

$$\begin{aligned} 2(x|y)_o &= \mathrm{d}(x, o) + \mathrm{d}(o, y) - \mathrm{d}(x, y) = \mathrm{d}(x, o) + \mathrm{d}(o, y) - \mathrm{d}(x, u) - \mathrm{d}(u, y) \\ &\le 2\,\mathrm{d}(u, o). \end{aligned}$$

So $(x|y)_o \le \mathrm{d}(o, \ell)$.

Let $\ell_x$ be a geodesic segment joining $o$ to $x$ and let $\ell_y$ be a geodesic segment joining $o$ to $y$. Since $\ell \subset \mathcal{N}_\delta(\ell_x) \cup \mathcal{N}_\delta(\ell_y)$ and $\ell$ is connected, there exists a point $u \in \ell$ such that

$$u \in \mathcal{N}_\delta(\ell_x) \cap \mathcal{N}_\delta(\ell_y).$$

Pick $x' \in \ell_x$ and $y' \in \ell_y$ with

$$\mathrm{d}(x', u), \mathrm{d}(y', u) < \delta.$$

Then

$$\begin{aligned}2(x|y)_o &= \mathrm{d}(x,o) + \mathrm{d}(o,y) - \mathrm{d}(x,y)\\ &= \mathrm{d}(x,x') + \mathrm{d}(x',o) + \mathrm{d}(o,y') + \mathrm{d}(y',y) - \mathrm{d}(x,u) - \mathrm{d}(u,y)\\ &> -2\delta + \mathrm{d}(x',o) + \mathrm{d}(o,y') \geq -4\delta + 2\,\mathrm{d}(o,u).\end{aligned}$$

Hence $(x|y)_o > \mathrm{d}(o,\ell) - 2\delta$.

(2). If $\mathrm{d}(\ell_x(t), \ell_y(t)) \leq 2\delta$, then

$$\begin{aligned}2(x|y)_o &= \mathrm{d}(x,o) + \mathrm{d}(o,y) - \mathrm{d}(x,y)\\ &= 2t + \mathrm{d}(x,\ell_x(t)) + \mathrm{d}(\ell_y(t),y) - \mathrm{d}(x,y)\\ &\geq 2t - 2\delta.\end{aligned}$$

So $(x|y)_o \geq T - \delta$.

If $(x|y)_o \leq \delta$, then $(x|y)_o \leq \delta + T$. So we can assume that $(x|y)_o > \delta$. Then fix $t := (x|y)_o - \delta$. Let $\ell_{xy}$ be a geodesic joining $x$ to $y$. Then there exists $v \in \ell_y \cap \ell_{xy}$ such that

$$\mathrm{d}(\ell_x(t), v) < \delta.$$

We claim that $v \in \ell_y$. If not, then $v \in \ell_{xy}$ and

$$\mathrm{d}(o,\ell_{xy}) \leq \mathrm{d}(o,v) < \mathrm{d}(o,\ell_x(t)) + \delta = t + \delta = (x|y)_0 \leq \mathrm{d}(o,\ell_{xy})$$

and so we have a contradiction. So $v = \ell_y(s)$ for some $s \geq 0$. Then

$$|t-s| \leq |\mathrm{d}(o,\ell_x(t)) - \mathrm{d}(o,\ell_y(s))| \leq \mathrm{d}(\ell_x(t),\ell_y(s)) < \delta$$

and so

$$\mathrm{d}(\ell_x(t),\ell_y(t)) \leq |t-s| + \mathrm{d}(\ell_y(s),\ell_y(t)) < 2\delta.$$

Thus $T \geq t = (x|y)_o - \delta$.

□

## 36. Establishing Ancona's estimates

Suppose $\Gamma$ is a non-elementary word hyperbolic group, $S$ is a finite symmetric generating set, $X$ is the associated Cayley graph, and $\mathrm{d}_S$ is the associated word metric on $\Gamma$. Let $\mu$ be a finitely supported probability measure on $\Gamma$ such that the associated random walk is irreducible.

In this section we verify Ancona's estimates and hence show that the Martin and Gromov boundary coincide. The arguments that follow come from recent work of Gekhtman–Gerasimov–Potyagailo–Yang [GGPY21] concerning the Martin boundary of relatively hyperbolic groups.

Given a finite list $\tau = (\tau_1, \ldots, \tau_n)$ of elements of $\Gamma$, let

$$\omega(\tau) := \begin{cases}\mu(\tau_1^{-1}\tau_2)\cdots\mu(\tau_{n-1}^{-1}\tau_n) & \text{if } n > 1\\ 1 & \text{if } n = 1.\end{cases}$$

A list is *admissible* if $\omega(\tau) > 0$, i.e. $\tau_j^{-1}\tau_{j+1} \in \mathrm{supp}(\mu)$ for all $j$.

For $x, y \in \Gamma$, let $P(x,y)$ be the set of finite lists starting at $x$ and ending at $y$. Then

$$G(x,y) = \sum_{\tau \in P(x,y)} \omega(\tau).$$

Hence we can define a measure $\Pr_{x,y}$ on $P(x,y)$ by

$$\Pr_{x,y}(A) = \frac{1}{G(x,y)} \sum_{\tau \in A} \omega(\tau).$$

**Theorem 36.1.** *There exists $R_0 > 0$ such that: if $x, y \in \Gamma$ and $z \in \Gamma$ is in a geodesic segment joining $x$ to $y$, then*

$$\Pr_{x,y}(\tau : \tau \cap B_{R_0}(z) \neq \emptyset) > 1/2.$$

As a corollary, we can verify Ancona's estimates.

**Corollary 36.2.** *For every $R > 0$ there exists $C = C(R) > 1$ such that if $x, y, z \in \Gamma$ and $z$ is in the $R$-neighborhood of a geodesic segment joining $x$ to $y$, then*

$$G(x,y) \leq C G(x,z) G(z,y).$$

*Hence by Proposition 34.1, the identity map $\Gamma \to \Gamma$ extends to a homeomorphism $\overline{\Gamma}^M \to \overline{\Gamma}^{\mathcal{G}}$.*

*Proof of Corollary.* By the theorem

$$\begin{aligned} G(x,y) &\leq 2 \sum_{\substack{\tau \in P(x,y) \\ \tau \cap B_{R+R_0}(z) \neq \emptyset}} \omega(\tau) \leq 2 \sum_{w \in B_{R+R_0}(z)} \sum_{\tau_1 \in P(x,w)} \sum_{\tau_2 \in P(w,y)} \omega(\tau_1)\omega(\tau_2) \\ &= 2 \sum_{w \in B_{R+R_0}(z)} G(x,w) G(w,y). \end{aligned}$$

Now, $w \mapsto G(w,y)$ is $\mu$-superharmonic and $w \mapsto G(x,w)$ is $\hat{\mu}$-superharmonic, where $\hat{\mu}(g) = \mu(g^{-1})$. Hence by Harnack's inequality, there exists $C_1 = C_1(R + R_0) > 0$ such that

$$G(x,y) \leq 2C_1 \sum_{w \in B_{R+R_0}(z)} G(x,z)G(z,y) = 2C_1 \left| B_{R+R_0}(z) \right| G(x,z)G(z,y).$$

Since the word metric is $\Gamma$-invariant, $|B_{R+R_0}(z)|$ only depends on $R + R_0$ (and not $z$).

□

36.1. **Proof of Theorem 36.1.** The idea is to show that long paths have low probability and show that paths that do not intersect $B_R(z)$ must be long.

**Lemma 36.3.** *There exist $0 < \phi < 1$ and $A > 0$ such that: if $x \neq y$, then*

$$\Pr_{x,y}(\tau : \text{Length}(\tau) \geq M) \leq \phi^{M - A\, \mathrm{d}_S(x,y)}.$$

*Proof.* For $t \in \mathbb{R}$ consider the (possibly divergent) power series

$$G_t(x,y) := \sum_{n=0}^{\infty} t^n p_n(x,y).$$

Recall from Example 26.4 that $\Gamma$ is non-amenable and so by Theorem 26.12 the spectral radius satisfies

$$\rho(\mu) < 1.$$

Further, the spectral radius satisfies

$$\rho(\mu) := \limsup_{n \to \infty} p_n(x,y)^{1/n}$$

(see Observation 26.9). Then $G_t$ converges when $|t| < \rho(\mu)^{-1}$ and for such $t$ the function $x \mapsto G_t(x, y)$ is superharmonic (check!).

Fix $t \in (1, \rho(\mu)^{-1})$. Then

$$\begin{aligned}\Pr_{x,y}(\tau : \text{Length}(\tau) \geq M) &= \frac{1}{G(x,y)} \sum_{n=M}^{\infty} p_n(x,y) \leq \frac{1}{G(x,y)} t^{-M} \sum_{n=M}^{\infty} t^n p_n(x,y) \\ &\leq \frac{G_t(x,y)}{G(x,y)} t^{-M}.\end{aligned}$$

Then, by Harnack's inequality, there exists $\lambda > 1$ such that

$$\Pr_{x,y}(\tau : \text{Length}(\tau) \geq M) \leq \frac{G_t(y,y)}{G(y,y)} \lambda^{2\,\mathrm{d}_S(x,y)} t^{-M} \leq \left( \frac{G_t(y,y)}{G(y,y)} \lambda^2 \right)^{\mathrm{d}_S(x,y)} t^{-M}.$$

Now let $\phi := 1/t$ and $A := -\log_t \left( \frac{G_t(y,y)}{G(y,y)} \lambda^2 \right)$. □

Fix $\epsilon > 0$, $\mathrm{D}_v$, and $\overline{\mathrm{D}}_v$ as in Section 35. Then there exists $C > 1$ such that

$$C^{-1} e^{-\epsilon (x|y)_{\mathrm{id}}} \leq \mathrm{D}_v(x,y) \leq C e^{-\epsilon (x|y)_{\mathrm{id}}}$$

for all $x, y \in X$.

For $r > 0$ let

$$E_r(x) := \left\{ y \in \Gamma : \mathrm{d}_S(\mathrm{id}, y) < r \text{ and } \overline{\mathrm{D}}_v(x,y) < \frac{1}{r} \right\}.$$

Then let $Q_r(x,y) \subset P(x,y)$ be the set of admissible lists that intersect both of $E_{2r}(x)$, $E_{2r}(y)$ but do not intersect at least one of $E_r(x), E_r(y)$.

**Observation 36.4.** If $x, y \in \Gamma$ are distinct, then

$$\{\tau \in P(x,y) : \tau \text{ admissible and } \tau \cap B_R(\mathrm{id}) = \emptyset\} \subset \bigcup_{m=0}^{\infty} Q_{2^m R}(x,y).$$

*Proof.* Fix $\tau \in P(x,y)$ admissible with $\tau \cap B_R(\mathrm{id}) = \emptyset$. Then

$$\tau \cap (E_R(x) \cup E_R(y)) \subset \tau \cap B_R(\mathrm{id}) = \emptyset.$$

Further, $x \in E_{2^m R}(x)$ and $y \in E_{2^m R}(y)$ when $m$ is sufficiently large. So there exists a minimal $m \geq 0$ such that

$$\tau \cap E_{2^{m+1} R}(x) \neq \emptyset \quad \text{and} \quad \tau \cap E_{2^{m+1} R}(x) \neq \emptyset$$

and with this choice $\tau \in Q_{2^m R}(x,y)$. Thus the first assertion is true.

□

**Lemma 36.5.** *There exists a function $h : (0, \infty) \to [0, \infty)$ such that*

1. $\lim_{r \to \infty} \frac{h(r)}{r} = +\infty$
2. *If $x, y \in \Gamma$ are distinct,* id *is contained in a geodesic joining $x$ to $y$, $u \in E_{2r}(x)$, and $v \in E_{2r}(y)$, then any admissible list joining $u$ to $v$ that does not intersect $E_r(x)$ or $E_r(y)$ has length at least $h(r)$.*

*Proof.* Since id is contained in a geodesic joining $x$ to $y$, we have

$$(x|y)_{\mathrm{id}} = \frac{1}{2} \left( \mathrm{d}(x, \mathrm{id}) + \mathrm{d}(\mathrm{id}, y) - \mathrm{d}(x,y) \right) = 0$$

Further, $x \neq y$ and so

$$\mathrm{d}_S(x,y) \geq 1.$$

Then

$$\overline{\mathrm{D}}_v(x,y) = \min\left\{\epsilon\,\mathrm{d}_S(x,y), C^{-1}e^{-\epsilon(x|y)_{\mathrm{id}}}\right\} \geq \min\{\epsilon, C^{-1}\}.$$

Further, by symmetry, it suffices to consider an admissible list $\tau$ joining $u$ to $v$ that does not intersect $E_r(x)$.

**Case 1:** Assume $\tau \cap B_r(\mathrm{id}) = \emptyset$. Let $L$ be a path in the Cayley graph obtained by joining successive elements in $\tau$ by a geodesic. Let $T := \max_{g\in\mathrm{supp}(\mu)} |g|_S$. Then $L \cap B_{r-T}(\mathrm{id}) = \emptyset$ and $\mathrm{Length}(L) \leq T\mathrm{Length}(\tau)$.

If $L = (\gamma_1, \ldots, \gamma_m)$, then

$$\begin{aligned}\overline{\mathrm{D}}_v(\gamma_j,\gamma_{j+1}) \leq \mathrm{D}_v(\gamma_j,\gamma_{j+1}) &\leq Ce^{-\epsilon(\gamma_j|\gamma_{j+1})_{\mathrm{id}}} = Ce^{-\epsilon/2(\mathrm{d}_S(\gamma_j,\mathrm{id})+\mathrm{d}_S(\mathrm{id},\gamma_{j+1})-\mathrm{d}_S(\gamma_j,\gamma_{j+1})}\\ &\leq Ce^{-\frac{\epsilon}{2}(r-T+r-T-1)} \leq C_2e^{-\epsilon r}\end{aligned}$$

where $C_2 := Ce^{\epsilon T+\epsilon/2}$. Further,

$$\overline{\mathrm{D}}_v(u,v) \geq \overline{\mathrm{D}}_v(x,y) - \overline{\mathrm{D}}_v(x,u) - \overline{\mathrm{D}}_v(v,y) \geq \min\{\epsilon, C^{-1}\} - 2\frac{1}{2r}.$$

So

$$\min\{\epsilon, C^{-1}\} - \frac{1}{r} \leq \overline{\mathrm{D}}_v(u,v) \leq \sum_{j=1}^{m} \overline{\mathrm{D}}_v(\gamma_j,\gamma_{j+1}) \leq C_2\mathrm{Length}(L)e^{-\epsilon r} \leq C_2T\mathrm{Length}(\tau)e^{-\epsilon r}.$$

Thus

$$\mathrm{Length}(\tau) \geq \left(\min\{\epsilon, C^{-1}\} - \frac{1}{r}\right) C_2^{-1}T^{-1}e^{\epsilon r}.$$

**Case 2:** Assume $\tau \cap B_r(\mathrm{id}) \neq \emptyset$. Let $u' \in \tau$ be the first point of intersection of $\tau$ with $B_r(\mathrm{id})$. Since $u' \notin E_r(x)$, we have $\overline{\mathrm{D}}_v(u',x) \geq \frac{1}{r}$. Hence

$$\overline{\mathrm{D}}_v(u,u') \geq \overline{\mathrm{D}}_v(u',x) - \overline{\mathrm{D}}_v(u,x) \geq \frac{1}{r} - \frac{1}{2r} = \frac{1}{2r}.$$

Then, arguing as in Case 1,

$$\frac{1}{2r} \leq \overline{\mathrm{D}}_v(u,u') \leq C_2T\mathrm{Length}(\tau)e^{-\epsilon r}.$$

Thus

$$\mathrm{Length}(\tau) \geq \frac{1}{2r}C_2^{-1}T^{-1}e^{\epsilon r}.$$

Hence in all cases, $\tau$ has length at least

$$h(r) = \max\left\{0, \min\left\{\min\{\epsilon, C^{-1}\} - \frac{1}{r}, \frac{1}{2r}\right\}\right\} C_2^{-1}T^{-1}e^{\epsilon r}.$$

When $r$ is sufficiently large,

$$h(r) := C_2^{-1}T^{-1}\frac{e^{\epsilon r}}{2r}$$

and so $\lim_{r\to\infty} \frac{h(r)}{r} = +\infty$. □

**Lemma 36.6.** *If $x, y \in \Gamma$ are distinct and* id *is contained in a geodesic joining $x$ to $y$, then*

$$\mathrm{Pr}_{x,y}(Q_r(x,y)) \leq \phi^{h(r)-4Ar}.$$

*Proof.* Let $Q'_r(x,y) \subset Q_r(x,y)$ denote the lists that intersect $E_{2r}(x)$ before $E_{2r}(y)$. Then for $\tau \in Q'_r(x,y)$ we can write $\tau = (\tau_1, u, \tau_2, v, \tau_3)$ where $\tau_1 \cap (E_r(x) \cup E_{2r}(y)) = \emptyset$, $u \in E_{2r}(x)$, $v \in E_{2r}(y)$, and $\tau_3 \cap E_{2r}(y) = \emptyset$. Since $\tau$ does not intersect $E_r(x)$ or $E_r(y)$, we also have

$$\mathrm{Length}(u, \tau_2, v) \geq h(r)$$

and since $u, v \in E_{2r}(x) \cup E_{2r}(y) \subset B_{2r}(\mathrm{id})$, we have $\mathrm{d}_S(u,v) \leq 4r$.

Then

$$\Pr_{x,y}(Q'_r(x,y)) = \frac{1}{G(x,y)} \sum_{\tau=(\tau_1,u,\tau_2,v,\tau_3) \in Q'_r(x,y)} \omega(\tau_1, u)\tau(u, \tau_2, v)\tau_3(v, \tau_3)$$

Let $P'(x,y)$ be the lists that intersect both $E_{2r}(x)$ and $E_{2r}(y)$, and intersect $E_{2r}(x)$ before $E_{2r}(y)$. For $\tau \in P'(x,y)$, we can write $\tau = (\tau_1, u, \tau_2, v, \tau_3)$ in the same way except that $\tau_2$ has no constraints. Thus by Lemmas 36.3 and 36.5,

$$\Pr_{x,y}(Q'_r(x,y)) \leq \phi^{h(r)-4Ar} \Pr_{x,y}(P'(x,y)).$$

The lemma then follows after a similar analysis applied to the paths intersecting $E_{2r}(y)$ before $E_{2r}(x)$. □

Fix $R_0 > 1$ such that

$$\frac{h(R_0 t)}{R_0 t} \geq 4A + 2$$

for all $t \geq 1$ and

$$\frac{\phi^{R_0}}{1 - \phi^{R_0}} < \frac{1}{2}.$$

**Lemma 36.7.** *If $x, y \in \Gamma$ and $z$ is contained in a geodesic joining $x$ to $y$, then*

$$\Pr_{x,y}(\tau \in P(x,y) : \tau \cap B_{R_0}(z) \neq \emptyset) < \frac{1}{2}.$$

*Proof.* By translating by $z^{-1}$, we can assume that $z = \mathrm{id}$. Then by the previous lemmas,

$$\begin{aligned}\Pr_{x,y}(\tau \in P(x,y) : \tau \cap B_{R_0}(\mathrm{id}) \neq \emptyset) &\leq \sum_{m=0}^{\infty} \Pr_{x,y}(Q_{2^m R_0}(x,y)) \\ &\leq \sum_{m=0}^{\infty} \phi^{h(2^m R_0) - 2^{m+2} A R_0} \leq \sum_{m=0}^{\infty} \phi^{2^{m+1} R_0} \leq \sum_{m=0}^{\infty} \phi^{(m+1)R_0} = \frac{\phi^{R_0}}{1 - \phi^{R_0}} < \frac{1}{2}. \square\end{aligned}$$

## References


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